Reversing Subdivision Rules: Local Linear Conditions and Observations on Inner Products

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Abstract

In a previous work [32] we investigated how to reverse subdivision rules using global least squares fitting. This led to multiresolution structures that could be viewed as semiorthogonal wavelet systems whose inner product was that for finite-dimensional Cartesian vector space. We produced simple and sparse reconstruction filters, but had to appeal to matrix factorization to obtain an efficient, exact decomposition. We also made some observations on how the inner product that defines the semiorthogonality influences the sparsity of the reconstruction filters.

In this work we carry the investigation further by studying biorthogonal systems based upon subdivision rules and local least squares fitting problems that reverse the subdivision. We are able to produce multiresolution structures for some common univariate subdivision rules that have both sparse reconstruction and decomposition filters. Three will be presented here – for quadratic and cubic B-spline subdivision and for the 4-point interpolatory subdivision of Dyn et. al. We observe that each biorthogonal system we produce can be interpreted as a semiorthogonal system with an inner product induced on the multiresolution that is quite different from that normally used. Some examples of the use of this approach on images, curves, and surfaces. are given.

keywords: subdivision, least squares, wavelets, curves, surfaces, multiresolution.

1 Setting

Assume that the points for a curve or surface, or the pixels for an image, are given. Denote them by c_ℓ^{k+1}. They will be referred to as the *fine data*. In some circumstances this data could have been produced by subjecting a smaller number of points or pixels, the *coarse data*, to a *subdivision process*:

$$c_\ell^{k+1} = \sum_{\lambda \in \ddot{\mathcal{P}}_\ell^{k+1}} p_{\ell,\lambda}^{k+1} c_\lambda^k \tag{1.1}$$

In other circumstances the data are simply given, but it may be of interest to find an approximate set of coarse data c_λ^k which, when subjected to the subdivision process defined by the $p_{\ell,\lambda}^{k+1}$, almost reproduce the fine data; that is, via the *reconstruction filter*

$$c_\ell^{k+1} = \sum_{\lambda \in \ddot{\mathcal{P}}_\ell^{k+1}} p_{\ell,\lambda}^{k+1} c_\lambda^k + \sum_{\kappa \in \ddot{\mathcal{Q}}_\ell^{k+1}} q_{\ell,\kappa}^{k+1} d_\kappa^k \tag{1.2}$$

The second summation represents the amount by which a subdivision of the coarse points C^k fails to reproduce the fine points C^{k+1}.

This the the *multiresolution* setting, for if the approximation process is repeated a number of times starting with the c_λ^k, the originally given data c_ℓ^{k+1} will break down into a small amount of *coarse* data c_λ^{k-N} and layers of *detail information* $d_\kappa^{k-N}, \ldots, d_\kappa^k$ from which the original data can be recovered using the *reconstruction coefficients* $\{p_{\ell,\lambda}^j : \lambda \in \ddot{\mathcal{P}}_\ell^j, j = k-N+1, \ldots, k+1\}$ and $\{q_{\ell,\kappa}^j : \kappa \in \ddot{\mathcal{Q}}_\ell^j, j = k-N+1, \ldots, k+1\}$. In the best of situations, the coarse data and the detail information can be generated by the use of a simple set of *decomposition filters*:

$$c_\lambda^k = \sum_{\ell \in \dot{\mathcal{A}}_\lambda^{k+1}} a_{\lambda,\ell}^{k+1} c_\ell^{k+1} \tag{1.3}$$

$$d_\kappa^k = \sum_{\ell \in \dot{\mathcal{B}}_\kappa^{k+1}} b_{\kappa,\ell}^{k+1} c_\ell^{k+1} \tag{1.4}$$

given by the *analysis coefficients* $\{a_{\lambda,\ell}^j : \ell \in \dot{\mathcal{A}}_\lambda^j, j = k-N+1, \ldots, k+1\}$ and $\{b_{\kappa,\ell}^j : \ell \in \dot{\mathcal{B}}_\kappa^j, j = k-N+1, \ldots, k+1\}$. (Index sets for the first of a pair of indices are indicated by a single dot and those for the second of a pair by a double dot.)

Practical subdivision rules are *sparse*; that is, all but a finite number of $p_{\ell,\lambda}^j$ are nonzero for any fixed (ℓ, j) and also for any fixed (λ, j). Indeed for all j, and except possibly for a few ℓ associated with "boundary situations," it is usual for these nonzero coefficients to be an identical small set of numbers repeated for each ℓ. Moreover, the index locations λ for these repeated numbers shift in a regular way with each ℓ.

In multivariate cases, each subdivision rule (1.1) imposes its own *subdivision connectivity* on the points, reflected in the index sets $\ddot{\mathcal{P}}_\ell^{k+1}$. For example, a well-known surface subdivision rule due to Chaikin [5] applies to tensor-product surfaces. The index sets represented by $\ddot{\mathcal{P}}_\ell^{k+1}$ in (1.1) are sets of of pairs $\lambda = (\mu, \nu)$, $\ell = (\rho, \sigma)$ that associate each point with its four neighbors in a quadrilateral index lattice. Another surface subdivision rule due to Dyn, Levin, and Gregory [18, 39] requires triangular mesh connectivity; and so on.

In this study we shall restrict ourselves to univariate rules. Mesh connectivity simplifies to simple sequence adjacency: the geometry of curves. Of course, the results apply to *tensor-product* surfaces as well, in the standard way (see e.g. [34]). We shall leave the study of more general surface geometry and mesh connectivity for later.

We shall study how, given a subdivision rule defined by simple coefficients $p^j_{\ell,\lambda}$, we can sometimes define coefficients $a^j_{\lambda,\ell}$, $b^j_{\kappa,\ell}$, and $q^j_{\ell,\kappa}$, of comparable simplicity. The goal is to produce the coarse data c^k_λ as a good approximation to the fine data c^{k+1}_ℓ in the sense that the approximation is geometrically realistic and yields small errors, $q^{k+1}_{\ell,\kappa} d^k_\kappa$.

The results we present involve only operations with the quantities $A^{k+1} = \{a^{k+1}_{\lambda,i}\}$, $B^{k+1} = \{b^{k+1}_{\kappa,\ell}\}$, C^{k+1}, P^{k+1}, and Q^{k+1}. We start with given subdivision/reconstruction coefficients P^{k+1} and generate, in a self-contained way, the remaining reconstruction coefficients Q^{k+1} and the decomposition coefficients A^{k+1} and B^{k+1} by purely linear-algebraic means. Even though our construction will be self-contained, it is reasonable to make some reflections on issues normally associated with multiresolutions, namely *wavelet systems*. Behind every multiresolution there are nested, inner-product spaces $\cdots \subset \mathcal{V}^{k-1} \subset \mathcal{V}^k \subset \mathcal{V}^{k+1} \subset \cdots$ each spanned by a basis of *scale functions* ϕ^k_λ that have been chosen to have some desirable properties. The difference spaces $\mathcal{W}^k = \mathcal{V}^{k+1} - \mathcal{V}^k$, with $\mathcal{W}^k \cap \mathcal{V}^k = \{0\}$, are spanned by a basis of *wavelets* ψ^k_λ, again chosen for some properties. In our geometric setting these spaces are real, Hilbert spaces of finite dimension, which greatly simplifies most of what we shall be doing.

Because of the nesting,

$$\begin{aligned}
\phi^k_\lambda &= \sum_{\ell \in \dot{\mathcal{P}}^{k+1}_\lambda} p^{k+1}_{\ell,\lambda} \phi^{k+1}_\ell \\
\psi^k_\kappa &= \sum_{\ell \in \dot{\mathcal{Q}}^{k+1}_\kappa} q^{k+1}_{\ell,\kappa} \phi^{k+1}_\ell
\end{aligned} \tag{1.5}$$

where the coefficients $p^{k+1}_{\ell,\lambda}$ and $q^{k+1}_{\ell,\kappa}$ are as in (1.2). Conversely,

$$\phi^{k+1}_\ell = \sum_{\lambda \in \ddot{\mathcal{A}}^{k+1}_\ell} a^{k+1}_{\lambda,\ell} \phi^k_\lambda + \sum_{\kappa \in \ddot{\mathcal{B}}^{k+1}_\ell} b^{k+1}_{\kappa,\ell} \psi^k_\kappa \tag{1.6}$$

where the coefficients $a^{k+1}_{\lambda,\ell}$ and $b^{k+1}_{\kappa,\ell}$ are as in (1.3) and (1.4).

In the original work on multiresolutions, only those nested spaces were of interest for which each scale function in any of the $\mathcal{V}^j$ could be derived from a single function in one of the spaces, $\mathcal{V}^0$, through shifting and dilating (and similarly for the wavelets). Later, this was relaxed to *multiwavelets* where the bases for the spaces $\mathcal{V}^j$ and $\mathcal{W}^j$ were constructed via shifts and dilations applied variously to a small set of functions. Wavelet systems that depend on shifts and dilations have come to be known as *first generation wavelets*. Discarding the requirement that the basis functions be produced by shifts and dilations has led to the study of *second generation wavelets*.

For any of the above categories of wavelet systems, a system may be required to be *orthogonal* with respect to the inner product on the spaces; that is for any j, $\left\langle \phi^j_{\ell_1}, \phi^j_{\ell_2} \right\rangle$ would be nonzero if and only if $\ell_1 = \ell_2$, $\left\langle \psi^{j_1}_{\ell_1}, \psi^{j_2}_{\ell_2} \right\rangle$ would be nonzero if and only if $\ell_1 = \ell_2$ and $j_1 = j_2$, and $\left\langle \phi^j_{\ell_1}, \psi^j_{\ell_2} \right\rangle$ would always be zero. Orthogonal systems were studied first. Later, the restrictions of complete orthogonality were relaxed to permit *semiorthogonal* systems, for which only $\left\langle \phi^j_{\ell_1}, \psi^j_{\ell_2} \right\rangle = 0$ is required (and in which the ψ functions are sometimes called *pre-wavelets*). Recently, the consideration of scale/wavelet systems was expanded into *biorthogonal* systems. In these systems there are *primal* scale and wavelet functions, ϕ^j_λ

and ψ^j_κ, *dual* scale and wavelet functions, $\tilde{\phi}^j_\lambda$ and $\tilde{\psi}^j_\kappa$, and orthogonality conditions:

$$\begin{aligned} \left\langle \phi^j_{\ell_1}, \tilde{\phi}^j_{\ell_2} \right\rangle &= \delta_{\ell_1,\ell_2} \\ \left\langle \psi^j_{\ell_1}, \tilde{\psi}^j_{\ell_2} \right\rangle &= \delta_{\ell_1,\ell_2} \\ \left\langle \phi^j_{\ell_1}, \tilde{\psi}^j_{\ell_2} \right\rangle &= 0 \\ \left\langle \psi^j_{\ell_1}, \tilde{\phi}^j_{\ell_2} \right\rangle &= 0 \end{aligned} \tag{1.7}$$

The dual scales and wavelets span the dual spaces $\tilde{\mathcal{V}}^j$ and $\tilde{\mathcal{W}}^j$, respectively. But the simplicity of our finite dimension setting allows us to make the identifications $\tilde{\mathcal{V}}^j \equiv \mathcal{V}^j$ and $\tilde{\mathcal{W}}^j \equiv \mathcal{W}^j$ and to write

$$\begin{aligned} \phi^k_\lambda &= \sum_{\ell \in \dot{\mathcal{P}}^{k+1}_\lambda} p^{k+1}_{\ell,\lambda} \phi^{k+1}_\ell \\ \psi^k_\kappa &= \sum_{\ell \in \dot{\mathcal{Q}}^{k+1}_\kappa} q^{k+1}_{\ell,\kappa} \phi^{k+1}_\ell \\ \tilde{\phi}^k_m &= \sum_{\ell \in \ddot{\tilde{\mathcal{A}}}^{k+1}_m} \tilde{a}^{k+1}_{m,\ell} \phi^{k+1}_\ell \\ \tilde{\psi}^k_n &= \sum_{\ell \in \ddot{\tilde{\mathcal{B}}}^{k+1}_n} \tilde{b}^{k+1}_{n,\ell} \phi^{k+1}_\ell \end{aligned} \tag{1.8}$$

Together, (1.7) and (1.8) imply

$$\left\langle \psi^{j_1}_{\ell_1}, \tilde{\psi}^{j_2}_{\ell_2} \right\rangle = \delta_{\ell_1,\ell_2} \delta_{j_1,j_2} \tag{1.9}$$

The relationship between $\tilde{A}^{k+1} = \{\tilde{a}^{k+1}_{m,\ell}\}$ and A^{k+1}, and between $\tilde{B}^{k+1} = \{\tilde{a}^{k+1}_{m,\ell}\}$ and B^{k+1}, will be given in Section 4. For more information on wavelets and multiresolution analysis see [6, 16, 25, 34].

2 Introduction and Other Work

In this paper we shall be working in a setting that is biorthogonal and second generation. Taking a given, sparse subdivision rule $p^{k+1}_{\ell,\lambda}$, we create from it sparse analysis coefficients $a^{k+1}_{\ell,\lambda}$ and $b^{k+1}_{\ell,\kappa}$, and we complete the system by creating the remaining sparse reconstruction coefficient $q^{k+1}_{\ell,\kappa}$. Implicitly, our construction induces an inner product. The inner product used is specific to each subdivision rule. Moreover, the coefficients A^{k+1}, B^{k+1}, and Q^{k+1} will be chosen to respect geometric (affine) issues.

There are several novelties in this setting. In the usual biorthogonal setting, all four sets of coefficients are to be constructed, including the subdivision coefficients P^{k+1} (alternatively, both primal and dual scales and wavelets are to be constructed). A large part of the problem becomes that of constructing wavelets with a certain number of vanishing moments in order to achieve a certain approximation power. In our setting, we choose to accept the given scale functions and their smoothness (by accepting the scale relationship given in the subdivision rule P^{k+1}), and we work from there.

In most of the wavelet literature, the inner product is chosen to be

$$\int_{-\infty}^{+\infty} f(t)\,\bar{g}(t)dt \tag{2.1}$$

which measures the underlying function spaces $\mathcal{V}^j$. This norm is immutable for all scale relationships. In our setting, we will not be specifying the inner product; it will be induced according to the scale relationship and some choices we make with regard to it.

The majority of the wavelet literature arises from the theory of approximation in a function space setting. Hence, constructions focus on the functions ϕ and ψ, their duals, and various transforms of these functions. Often, neither the scales nor wavelets, nor the coefficients A^{k+1}, B^{k+1}, P^{k+1}, and Q^{k+1}, are subjected to conditions desirable for images, curves, or surfaces. In our setting, as in the portion of the wavelet literature coming from graphics applications, such conditions should be observed; e.g., that primal scale functions should be nonnegative and partition unity and that only affine and vector combinations should be formed from the data points C^{k+1} [23]. More specifically, equation (1.2) specifies that the fine points C^{k+1} are expressed in terms of *affine combinations* (with coefficients P^{k+1}) of the coarse points C^k and *linear combinations* (with coefficients Q^{k+1}) of the detail information D^k, whose elements serve geometrically as *displacement vectors*. This means that the elements of P^{k+1} must satisfy

$$\sum_{\lambda \in \ddot{\mathcal{P}}_\ell^{k+1}} p_{\ell,\lambda}^{k+1} = 1 \tag{2.2}$$

for every ℓ. This is not a condition we need to impose here, since it must be true for any P^{k+1} that represents a subdivision. Moreover, there are no geometric conditions we need to impose on Q^{k+1}, since the elements of Q^{k+1} provide the coefficients of linear combinations of vectors. For equation (1.3) to be geometrically realistic, however, we must require

$$\sum_{\ell \in \dot{\mathcal{A}}_\lambda^{k+1}} a_{\lambda,\ell}^{k+1} = 1 \tag{2.3}$$

for every λ, since (1.3) expresses the coarse points C^k as affine combinations of the fine points C^{k+1}. Finally, for equation (1.4) to be geometrically realistic, we must require

$$\sum_{\ell \in \dot{\mathcal{B}}_\kappa^{k+1}} b_{\kappa,\ell}^{k+1} = 0 \tag{2.4}$$

since (1.4) represents a conversion of the points C^{k+1} into the vectors D^k.

Even in the graphics literature on wavelets, there is a focus on the underlying function space rather than the geometric data C^{k+1}. In our setting, although we shall try to remain aware of the underlying scales and wavelets, we shall be supplying a construction that is purely linear-algebraic in nature and based entirely on the given subdivision coefficients.

The most basic multiresolution analysis with sparse A^{k+1}, B^{k+1}, P^{k+1}, and Q^{k+1} is the Haar system [34]. This system is *orthogonal* in the inner product (2.1), but its approximation power does not extend beyond piecewise constants. Consequently its geometrical and image usefulness is limited.

A simple multiresolution system built up purely on the basis of local approximations among the data points, C^{k+1}, is due to Faber [20]. The construction used by Faber can be viewed as a very simple form of the construction we shall be using, although it is incomplete in that it was not developed in a way to provide analysis and reconstruction coefficients. Faber subdivision maps piecewise linear functions into piecewise linear functions; and as such, it is the first interesting subdivision rule for geometry, applying naturally to polyhedral mesh surfaces.

Using the inner product (2.1), sparse, first generation systems of higher approximation power have been produced. For example, orthogonal systems by Daubechies [13], and biorthogonal systems by Cohen, Daubechies, and Feauveau [10], Cohen [7], Herley and Vetterli [24], Karoui and Vaillancourt [26], to name but a few. However, most of the underlying scale functions are not entirely appropriate for curves, surfaces, and images, for which nonnegative functions that partition unity are desirable. Moreover, second generation systems are far more interesting in the context of graphics, since they offer the hope of constructing wavelets on domains and topologies that don't provide for shifts and dilations.

A powerful method for generating biorthogonal, second generation systems is via the *lifting construction* introduced by Sweldens [36]. He and Schröder use this construction to define wavelets on the surface of a sphere in [33] and show how to use lifting in a general setting in [37]. In the reverse direction, Daubechies and Sweldens [14] show how given wavelet systems can be decomposed into lifting steps. Hence, in this paper we are unlikely to achieve anything that could not be achieved via lifting. However, in usual practice, the lifting method constructs wavelets in a sequence of steps that are driven by goals formulated in terms of moments and the conventional inner product (2.1). One of the end products is the subdivision (scale) relationship given by the coefficients P^{k+1}. Our approach, as we have stated, will be to accept a given subdivision rule and to provide, by geometric and linear-algebraic considerations, the remainder of a system that is biorthogonal with respect to a norm that is induced by the subdivision.

There are a number of graphically-oriented papers that use the conventional inner product (2.1), together with one or another particular subdivision, to induce a wavelet system that establishes a multiresolution representation for curves and/or surfaces. Some of these papers focus on the creation of their representations for the purpose of *multiresolution display*, for example the work by Certain; et. al. [4], and that by Lounsbery, DeRose, and Warren [30]. Some papers encode the representation in a form that supports *multiresolution editing*, for example the work by Finkelstein and Salesin [21]. Other papers are interested in *compression*, for example the work by DeVore, Jawerth, and Lucier [15]. A particularly notable example to cite is the work by Reissell [31], which produces a sparse biorthogonal system of smooth, *symmetric*, *interpolating* scale functions. Reissell gives applications to compression, feature detection, and intersection location.

What we shall be adding is a construction that, so far on all univariate subdivisions investigated, has yielded primal and dual functions that are simpler than those given in these citations. Our approach has been carried out in a preliminary fashion to a non-tensor-product subdivision for polyhedral meshes. Further work is required here. Whether symmetry is supported depends upon the given subdivision rule. Multiresolution editing, while not covered in this paper, is achievable by organizing detail information into a format of local coordinate frames as proposed by [22].

Further afield in the graphics literature, Kobbelt; et. al. [28], create multiresolution meshes from a given mesh of *arbitrary connectivity*, which means that they forego the regularity present in the subdivision (and scale/wavelet) setting to handle much more flexible geometries. They encode their multiresolution information in a form suitable for multiresolution editing. This work is worth citing here, because their process of finding a coarser mesh from a finer one involves a local minimization. Their local minimization problems are solved iteratively at each data point, which is a computational price they pay for handling arbitrary connectivity. The concept of local minimization is one we also exploit, but the regularity of a subdivision setting allows us to solve the local problems universally and in advance.

A bridge between the material of [28] and that of the subdivision setting is provided by the work of Eck; et. al. [19], which introduces a means of approximating meshes with arbitrary connectivity by meshes with subdivision connectivity.

A vast number of theoretical papers can be found that address issues associating subdivision rules with multiresolutions and wavelet systems. As well the issue of *constructing* wavelets from subdivisions, issues worth mentioning here are those of *existence* and *stability*. The former issue asks when a basis of scales and wavelets for an infinite nesting of spaces in L^2 exists consistent with a subdivision. The second issue asks whether approximations of L^2 functions with respect to these basis functions can be trusted to have coefficients that are bounded in some way by the norms of the approximated functions (i.e., provide Riesz bases). Most of this theory rests in the realm of first generation wavelets. The setting of second generation wavelets that interests us is a less explored terrain for such issues. In any event, we are attempting only to be constructive here. It is our intention that the constructions be applied to well-established subdivisions (known to converge and yield multiresolutions in terms of the usual norm (2.1)), and we are concerned with practical graphics settings in which only a few levels of finite-dimensional, nested spaces are employed. Nevertheless, the following papers give the flavor of the issues. The paper by Cohen [8] takes one subdivision rule and discusses construction via the "cascade method". Cohen and Daubechies [9] discuss the issue of stability, and Dahmen and Micchelli [12] deal with existence.

Our paper is not the first to suggest that other inner products than (2.1) are possible for wavelet systems. In an earlier work [32] we used the Euclidian inner product on the data points C^{k+1} to construct wavelets of very small support, and Aldroubi, Eden, and Unser [2] have also investigated this inner product specifically for B-splines as scaling functions. Both of these papers deal with semiorthogonal systems. For biorthogonal systems, Sweldens [35] has investigated weighted inner products.

At the core of our construction lies a matrix problem: given a matrix $\mathbf{P}^{k+1}$ (with the subdivision coefficients P^{k+1} as entries), complete this to a square, nonsingular matrix and find a left inverse matrix

$$\begin{bmatrix} \mathbf{A}^{k+1} \\ \mathbf{B}^{k+1} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{k+1} \; \mathbf{Q}^{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{2.5}$$

from which the other coefficients Q^{k+1}, A^{k+1}, and B^{k+1} are obtained. The structure of the matrices $\mathbf{P}^{k+1}$ that arise from subdivisions is commonly *slanted*. This means that these matrices are banded, and the elements of all but possibly a few of the extreme left and right columns are a repeat of the elements of a previous column shifted down by two or more positions. In cyclic cases (for closed, periodic curves) this shifting pattern will wrap around; elements falling off the bottom of a column are reintroduced at the top, producing a *circulant* matrix. The structure of $\mathbf{P}^{k+1}$ should be carried over to $\mathbf{Q}^{k+1}$, $\mathbf{A}^{k+1}$, and $\mathbf{B}^{k+1}$. A paper that discusses a way of producing $\mathbf{Q}^{k+1}$ alone (to produce a semiorthogonal wavelet system) has been published by Lawton, Lee, and Shen [29]. Matrix extension techniques for the biorthogonal situation are given in the following papers: Dahmen and Micchelli employ matrix factorizations [11], Carnicer, Dahmen, and Peña [3] pay special attention to issues of stability, and Warren [38] uses lifting. In all cases, it is the conventional inner product (2.1) that is used, and the work by Warren is the only one to focus on the geometrical setting.

In the univariate case, for closed, periodic curves, an abstract formulation for the matrix problem, and its goal of producing simple $\mathbf{Q}^{k+1}$, $\mathbf{A}^{k+1}$, and $\mathbf{B}^{k+1}$ from a simple $\mathbf{P}^{k+1}$, can be given by asking when such a matrix has a slanted, circulant extension and inverse. The paper by Kautski and Turcajová [27] studies some of this issue.

Finally, in a work that is hard to fit into one of the foregoing categories yet is worth mentioning, Aldroubi, Abry, and Unser [1] take <u>two</u> subdivision rules and use them to build first generation biorthogonal systems, using one subdivision for the primal system and the other for the dual system.

In Section 3, which comprises the bulk of this paper, we shall carry out our construction in a specific case, using the subdivision for univariate, cubic B-splines. We do this to explore the issues of construction in a simple setting. In Section 4 we look at the matrix view of our construction and what that view says about the inner product on the underlying spaces. In Section 5 we summarize the construction in general. Section 6 will contain another B-spline subdivision rule for which the results of our construction were particularly pleasing. This section will also study a non-B-spline, interpolatory subdivision to show our approach in a different setting. We close in Section 7 with some example applications to curves, tensor-product surfaces, and images.

3 A Simple System: Cubic B-Splines

A simple setting is provided by cubic B-spline subdivision. A diagram illustrating this setting is given in Figure 1. This diagram is meant to show an interior section of a geometric figure that may be open or closed/periodic. We shall concentrate only on generic interior sections of subdivisions for simplicity in presentation. From (1.2) we know the relationship between the coarse points, the fine points, and the detail information. In order for (1.3) and (1.2) to be consistent, we must have:

$$\begin{aligned} c_\lambda^k &= \sum_{\ell \in \dot{\mathcal{A}}_\lambda^{k+1}} a_{\lambda,\ell}^{k+1} c_\ell^{k+1} \qquad (3.1) \\ &= \sum_{\mu \in \ddot{\mathcal{P}}_\ell^{k+1}} \left(\sum_{\ell \in \dot{\mathcal{A}}_\lambda^{k+1}} a_{\lambda,\ell}^{k+1} p_{\ell,\mu}^{k+1} \right) c_\mu^k + \sum_{\nu \in \ddot{\mathcal{Q}}_\ell^{k+1}} \left(\sum_{\ell \in \dot{\mathcal{A}}_\lambda^{k+1}} a_{\lambda,\ell}^{k+1} q_{\ell,\nu}^{k+1} \right) d_\nu^k \end{aligned}$$

implying

$$\sum_{\ell \in \dot{\mathcal{A}}_\lambda^{k+1}} a_{\lambda,\ell}^{k+1} p_{\ell,\mu}^{k+1} = \delta_{\lambda,\mu} \qquad (3.2)$$

and

$$\sum_{\ell \in \dot{\mathcal{A}}_\lambda^{k+1}} a_{\lambda,\ell}^{k+1} q_{\ell,\nu}^{k+1} = 0 \qquad (3.3)$$

(Note that (3.2) and (3.3) are represented by the top portion of (2.5).)

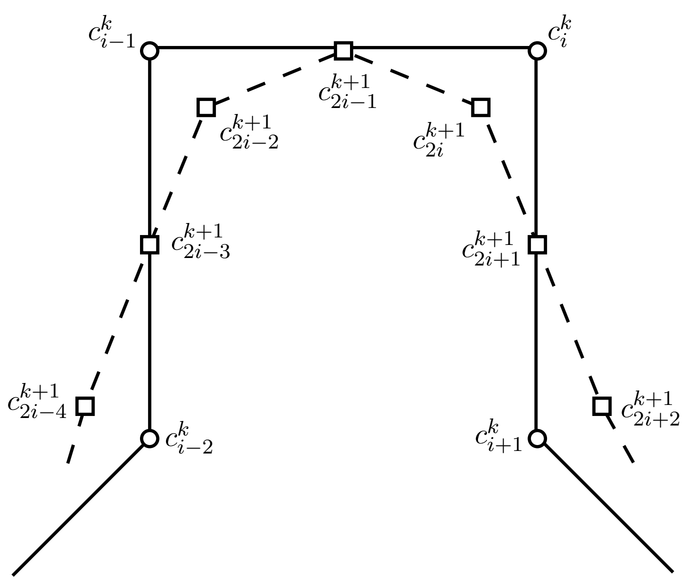


Figure 1: A diagram illustrating cubic B-spline subdivision

We shall construct the coefficients A^{k+1} to satisfy (3.2) in this Subsection and the following one, and we shall construct the coefficients Q^{k+1} to satisfy (3.3) in Subsection 3.4. This approach lets us ignore the second summation in (1.2) to construct A^{k+1} and concentrate purely on the first summation; that is, we can treat (1.1) as if it held even when C^{k+1} did not derive from C^k by subdivision.

According to (1.1), the relationships between the coarse and fine points in an interior section would be:

$$\begin{aligned}
c_{2i-3}^{k+1} &= \tfrac{1}{2}c_{i-2}^k + \tfrac{1}{2}c_{i-1}^k \\
c_{2i-2}^{k+1} &= \tfrac{1}{8}c_{i-2}^k + \tfrac{3}{4}c_{i-1}^k + \tfrac{1}{8}c_i^k \\
c_{2i-1}^{k+1} &= \tfrac{1}{2}c_{i-1}^k + \tfrac{1}{2}c_i^k \\
c_{2i}^{k+1} &= \tfrac{1}{8}c_{i-1}^k + \tfrac{3}{4}c_i^k + \tfrac{1}{8}c_{i+1}^k \\
c_{2i+1}^{k+1} &= \tfrac{1}{2}c_i^k + \tfrac{1}{2}c_{i+1}^k \\
c_{2i+2}^{k+1} &= \tfrac{1}{8}c_i^k + \tfrac{3}{4}c_{i+1}^k + \tfrac{1}{8}c_{i+2}^k \\
c_{2i+3}^{k+1} &= \tfrac{1}{2}c_{i+1}^k + \tfrac{1}{2}c_{i+2}^k
\end{aligned} \tag{3.4}$$

The indexing in these relationships has been chosen to point up the fact that the coarse point c_i^k has a natural correspondence with the fine point c_{2i}^{k+1}. A change in c_i^k most strongly influences c_{2i}^{k+1} in the subdivision. Conversely, we would expect any change in c_{2i}^{k+1} to have the most profound impact on c_i^k in any reasonable reversal of that subdivision.

In Subsection 3.1 we shall set local linear conditions that could define a set of analysis coefficients $a_{i,\lambda}^{k+1}$ that will produce c_i^k. In Subsection 3.2 we shall formulate a local least squares fitting problem to approximate c_i^k. In Subsection 3.3 we shall show how the two problems are equivalent. In Subsection 3.4 we shall find Q^{k+1} coefficients compatible with the A^{k+1} coefficients.

3.1 Finding the Coefficients A^{k+1} by Local Linear Equations

Equations (3.4) can be written as follows from the point of view of c_i^k:

$$\begin{aligned}
0c_i^k &= c_{2i-3}^{k+1} + K_{-3} \\
\tfrac{1}{8}c_i^k &= c_{2i-2}^{k+1} + K_{-2} \\
\tfrac{1}{2}c_i^k &= c_{2i-1}^{k+1} + K_{-1} \\
\tfrac{3}{4}c_i^k &= c_{2i}^{k+1} + K_0 \\
\tfrac{1}{2}c_i^k &= c_{2i+1}^{k+1} + K_1 \\
\tfrac{1}{8}c_i^k &= c_{2i+2}^{k+1} + K_2 \\
0c_i^k &= c_{2i+3}^{k+1} + K_3
\end{aligned} \tag{3.1.1}$$

where the symbols K hide the remaining terms of the equations in (3.4). Although we might be tempted to ignore the first and last of these equations in what follows, we shall see subsequently that we achieve more flexibility by including them. Indeed, in Subsection 3.2 we shall see that including equations even earlier and later in the sequence would make sense.

Each equation of (3.1.1) can be multiplied by a factor:

$$\frac{1}{8}a_{i,2i-2}^{k+1}c_i^k = a_{i,2i-2}^{k+1}c_{2i-2}^{k+1} + a_{i,2i-2}^{k+1}K_{-2}$$

and the results can be added

$$\begin{aligned}
&\frac{1}{8}a^{k+1}_{i,2i-2}c^k_i + \frac{1}{2}a^{k+1}_{i,2i-1}c^k_i + \frac{3}{4}a^{k+1}_{i,2i}c^k_i + \frac{1}{2}a^{k+1}_{i,2i+1}c^k_i + \frac{1}{8}a^{k+1}_{i,2i+2}c^k_i \\
&\quad = a^{k+1}_{i,2i-3}c^{k+1}_{2i-3} + a^{k+1}_{i,2i-2}c^{k+1}_{2i-2} + a^{k+1}_{i,2i-1}c^{k+1}_{2i-1} \\
&\qquad + a^{k+1}_{i,2i}c^{k+1}_{2i} + a^{k+1}_{i,2i+1}c^{k+1}_{2i+1} + a^{k+1}_{i,2i+2}c^{k+1}_{2i+2} + a^{k+1}_{i,2i+3}c^{k+1}_{2i+3} \\
&\qquad\quad + a^{k+1}_{i,2i-3}K^{k+1}_{-3} + a^{k+1}_{i,2i-2}K^{k+1}_{-2} + a^{k+1}_{i,2i-1}K^{k+1}_{-1} \\
&\qquad\qquad + a^{k+1}_{i,2i}K^{k+1}_{0} + a^{k+1}_{i,2i+1}K^{k+1}_{1} + a^{k+1}_{i,2i+2}K^{k+1}_{2} + a^{k+1}_{i,2i+3}K^{k+1}_{3}
\end{aligned} \tag{3.1.2}$$

The factors are taken to be those from the subdivision. They serve to weight the importance each of the c^k_j have in the creation of c^{k+1}_{2i} from the fine points.

This provides us with several conditions. Firstly, in order to produce the point c^k_i, we must have:

$$\frac{1}{8}a^{k+1}_{i,2i-2} + \frac{1}{2}a^{k+1}_{i,2i-1} + \frac{3}{4}a^{k+1}_{i,2i} + \frac{1}{2}a^{k+1}_{i,2i+1} + \frac{1}{8}a^{k+1}_{i,2i+2} = 1 \tag{3.1.3}$$

Secondly, since we would want to obtain c^k_i only from the fine points, we must have

$$\begin{aligned}
&a^{k+1}_{i,2i-3}K^{k+1}_{-3} + a^{k+1}_{i,2i-2}K^{k+1}_{-2} + a^{k+1}_{i,2i-1}K^{k+1}_{-1} \\
&\quad + a^{k+1}_{i,2i}K^{k+1}_{0} + a^{k+1}_{i,2i+1}K^{k+1}_{1} + a^{k+1}_{i,2i+2}K^{k+1}_{2} + a^{k+1}_{i,2i+3}K^{k+1}_{3} \\
&\quad = a^{k+1}_{i,2i-3}(-\frac{1}{2}c^k_{i-2} - \frac{1}{2}c^k_{i-1}) + a^{k+1}_{i,2i-2}(-\frac{1}{8}c^k_{i-2} - \frac{3}{4}c^k_{i-1}) \\
&\quad + a^{k+1}_{i,2i-1}(-\frac{1}{2}c^k_{i-1}) + a^{k+1}_{i,2i}(-\frac{1}{8}c^k_{i-1} - \frac{1}{8}c^k_{i+1}) \\
&\quad + a^{k+1}_{i,2i+1}(-\frac{1}{2}c^k_{i+1}) + a^{k+1}_{i,2i+2}(-\frac{3}{4}c^k_{i+1} - \frac{1}{8}c^k_{i+2}) + a^{k+1}_{i,2i+3}(-\frac{1}{2}c^k_{i+1} - \frac{1}{2}c^k_{i+2}) \\
&\quad = 0
\end{aligned} \tag{3.1.4}$$

Collecting terms of common coarse points together, this implies the following conditions:

$$\begin{aligned}
\tfrac{1}{2}a^{k+1}_{i,2i-3} + \tfrac{1}{8}a^{k+1}_{i,2i-2} &= 0 \\
\tfrac{1}{2}a^{k+1}_{i,2i-3} + \tfrac{3}{4}a^{k+1}_{i,2i-2} + \tfrac{1}{2}a^{k+1}_{i,2i-1} + \tfrac{1}{8}a^{k+1}_{i,2i} &= 0 \\
\tfrac{1}{8}a^{k+1}_{i,2i} + \tfrac{1}{2}a^{k+1}_{i,2i+1} + \tfrac{3}{4}a^{k+1}_{i,2i+2} + \tfrac{1}{8}a^{k+1}_{i,2i+3} &= 0 \\
\tfrac{1}{8}a^{k+1}_{i,2i+2} + \tfrac{1}{2}a^{k+1}_{i,2i+3} &= 0
\end{aligned} \tag{3.1.5}$$

Finally, in order to honor equation (2.3), we must have:

$$a^{k+1}_{i,2i-3} + a^{k+1}_{i,2i-2} + a^{k+1}_{i,2i-1} + a^{k+1}_{i,2i} + a^{k+1}_{i,2i+1} + a^{k+1}_{i,2i+2} + a^{k+1}_{i,2i+3} = 1 \tag{3.1.6}$$

All this can be summarized as the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_{i,2i-3}^{k+1} \\ a_{i,2i-2}^{k+1} \\ a_{i,2i-1}^{k+1} \\ a_{i,2i}^{k+1} \\ a_{i,2i+1}^{k+1} \\ a_{i,2i+2}^{k+1} \\ a_{i,2i+3}^{k+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.1.7}$$

The initial row of the matrix in (3.1.7) is the sum of the remaining rows. Similarly, the initial component of the right-hand side is the sum of the remaining components. Thus, the first equation is redundant. Any solution of the system

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_{i,2i-3}^{k+1} \\ a_{i,2i-2}^{k+1} \\ a_{i,2i-1}^{k+1} \\ a_{i,2i}^{k+1} \\ a_{i,2i+1}^{k+1} \\ a_{i,2i+2}^{k+1} \\ a_{i,2i+3}^{k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.1.8}$$

will automatically satisfy (3.1.6) and provide us with elements A^{k+1} that define an affine combination. Equation (3.1.7) exactly expresses equation (3.2) for $\lambda = i$.

This system of equations is underdetermined The general solution is given by

$$\begin{array}{rclcccc} a_{i,2i-3}^{k+1} & = & (\text{arbitrary}) & & & & \\ a_{i,2i-2}^{k+1} & = & -4a_{i,2i-3}^{k+1} & & & & \\ a_{i,2i-1}^{k+1} & = & 6a_{i,2i-3}^{k+1} & + & a_{i,2i+3}^{k+1} & - & \frac{1}{2} \\ a_{i,2i}^{k+1} & = & -4a_{i,2i-3}^{k+1} & - & 4a_{i,2i+3}^{k+1} & + & 2 \\ a_{i,2i+1}^{k+1} & = & a_{i,2i-3}^{k+1} & + & 6a_{i,2i+3}^{k+1} & - & \frac{1}{2} \\ a_{i,2i+2}^{k+1} & = & & & -4a_{i,2i+3}^{k+1} & & \\ a_{i,2i+3}^{k+1} & = & & & (\text{arbitrary}) & & \end{array} \tag{3.1.9}$$

The effect of removing the first and last of equations (3.1.1) can be achieved by setting $a_{i,2i-3}^{k+1} =$

$a^{k+1}_{2,2i+3} = 0$. This produces the solution:

$$\begin{array}{ccccccc} a^{k+1}_{i,2i-3} & a^{k+1}_{i,2i-2} & a^{k+1}_{i,2i-1} & a^{k+1}_{i,2i} & a^{k+1}_{i,2i+1} & a^{k+1}_{i,2i+2} & a^{k+1}_{i,2i+3} \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & 0 \end{array} \tag{3.1.10}$$

This solution corresponds to the system (3.1.8) with the variables $a^{k+1}_{i,2i-3}$ and $a^{k+1}_{i,2i+3}$, and correspondingly the first and last columns of the matrix, removed.

This diminished system is square and nonsingular, so the solution (3.1.10) is unique. However, as a crude estimate, if we were to use this solution as analysis coefficients, successive applications could affect c^{k+1}_{2i} in an unstable way: $c^{k+1}_{2i} \rightsquigarrow c^k_i \approx 2c^{k+1}_{2i} \rightsquigarrow c^k_{\frac{i}{2}} \approx 2^2 c^{k+1}_{2i} \rightsquigarrow \cdots \rightsquigarrow 2^j c^{k+1}_{2i}$. More broadly, the Euclidian norm of this solution is $\sqrt{4.5} \approx 2.12132$, and we might crudely expect successive "coarsenings" of the points c^{k+1} to be subjected to successive convolutions whose norms behave like powers of $\sqrt{4.5}$. These thoughts don't necessarily have any mathematical validity, but they do seem to correlate to what we shall be observing in Figures 2 and 3. This leads to the idea of finding the minimum norm solution of (3.1.9), which is given by

$$\begin{array}{ccccccc} a^{k+1}_{i,2i-3} & a^{k+1}_{i,2i-2} & a^{k+1}_{i,2i-1} & a^{k+1}_{i,2i} & a^{k+1}_{i,2i+1} & a^{k+1}_{i,2i+2} & a^{k+1}_{i,2i+3} \\ \hline \frac{23}{196} & -\frac{23}{49} & \frac{9}{28} & \frac{52}{49} & \frac{9}{28} & -\frac{23}{49} & \frac{23}{196} \end{array} \tag{3.1.11}$$

The norm of this solution is ≈ 1.34202. A comparison between the solutions represented by (3.1.10) and (3.1.11) is given by Figures 2 and 3. The dark curve plots 512 points measured around the coastline of an island in Norway, whose data was kindly provided by Morten Dæhlen. We used 3 applications of analysis coefficients to obtain 256, 128, and 64 points in sequence. The light curve connects the 64 points obtained as a result.

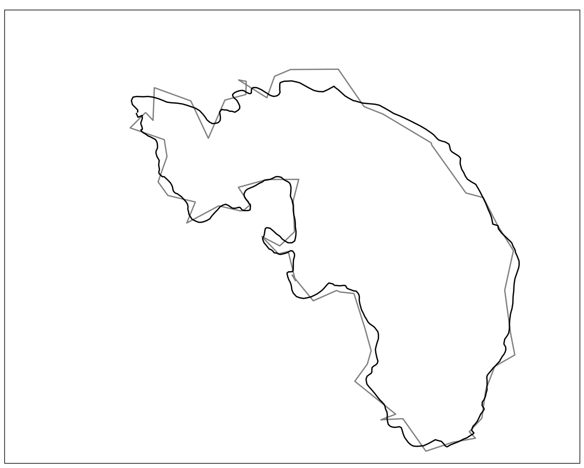


Figure 2: 64 coarse points from 512 data points using (3.1.10)

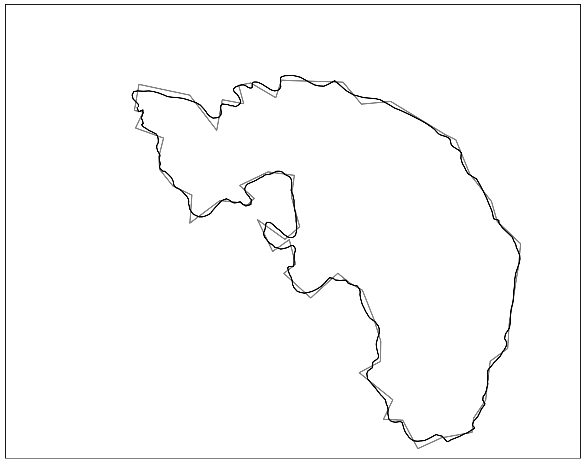


Figure 3: Using (3.1.11) (optimal 7-element optimal analysis coefficients)

Several other possibilities, both symmetric and antisymmetric, are:

$$\begin{array}{ccccccc} \underline{a^{k+1}_{i,2i-3}} & \underline{a^{k+1}_{i,2i-2}} & \underline{a^{k+1}_{i,2i-1}} & \underline{a^{k+1}_{i,2i}} & \underline{a^{k+1}_{i,2i+1}} & \underline{a^{k+1}_{i,2i+2}} & \underline{a^{k+1}_{i,2i+3}} \\ \frac{1}{9} & -\frac{4}{9} & \frac{5}{18} & \frac{10}{9} & \frac{5}{18} & -\frac{4}{9} & \frac{1}{9} \\ \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} \\ \frac{1}{6} & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{4} & 1 & 1 & -1 & \frac{1}{4} \end{array} \tag{3.1.12}$$

The first two of these possibilities are nearly optimal in the sense that their norms are only slightly different from that of (3.1.11). The solution that begins and ends with $\frac{1}{8}$ is particularly interesting, since these coefficients, like the coefficients P^{k+1} for the subdivision, involve only divisions by powers of 2, which means that they could be implemented very efficiently on silicon, in integer arithmetic using shifts. These coefficients were used to provide Figure 4. The solution that begins and ends with $\frac{1}{6}$ was the result of seeking a solution that had simple elements all less than 1 in magnitude. The norm of this solution is significantly larger than the optimal one, and its results on the island data are worse. Finally, an exploration on whether equally simple coefficients existed that had "support" equal to that of the subdivision coefficients yielded the asymmetric solution given last in (3.1.12). The results for the island data using these last two solutions are shown in Figure 5.

In geometric terms, we are deciding that c^{k+1}_{2i} has the closest association with c^k_i of any of the fine points in terms of the subdivision mask, and then we are testing masks of analysis coefficients that involve c^{k+1}_{2i} and its cohorts in wider and wider neighborhoods with respect to the subdivision connectivity. In doing so, we have found masks of analysis coefficients having 3, 5, and 7 elements. We also notice a correlation between the norm of the solution and the extent to which the coarse points track the original data.

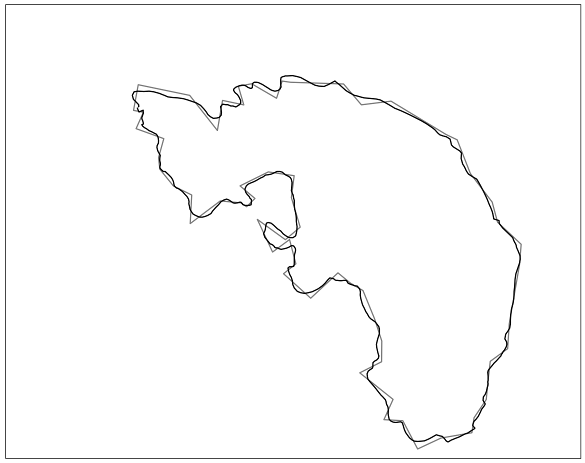
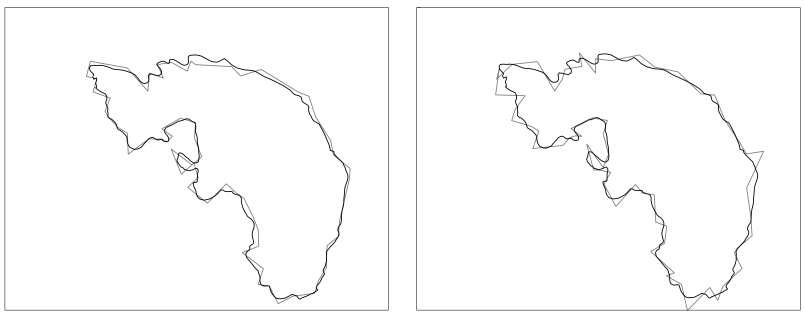


Figure 4: 64 coarse points using $\frac{1}{8}, -\frac{1}{2}, \frac{3}{8}, 1, \frac{3}{8}, -\frac{1}{2}, \frac{1}{8}$ (nearly optimal)



$\frac{1}{6}, -\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{6}$ $\quad$ $0, 0, -\frac{1}{4}, 1, 1, -1, \frac{1}{4}$

Figure 5: Nearly level and asymmetric coefficients

3.2 Finding the Coefficients $\mathbf{A}^{k+1}$ by Local Least Squares Fitting

Another way of obtaining c_i^k would be through direct approximation. Returning to (3.4), let us use least squares to estimate c_i^k and some of its neighboring coarse points. We shall then retain only the estimate of c_i^k, using similar least squares estimates to provide for the other coarse points. As an example, a matrix format for least squares using the equations of (3.4) would be:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c^k_{i-2} \\ c^k_{i-1} \\ c^k_{i} \\ c^k_{i+1} \\ c^k_{i+2} \end{bmatrix} \approx \begin{bmatrix} c^{k+1}_{2i-3} \\ c^{k+1}_{2i-2} \\ c^{k+1}_{2i-1} \\ c^{k+1}_{2i} \\ c^{k+1}_{2i+1} \\ c^{k+1}_{2i+2} \\ c^{k+1}_{2i+3} \end{bmatrix} \tag{3.2.1}$$

The solution, via the normal equations, is given by

$$\begin{bmatrix} \frac{191}{84} & -\frac{23}{21} & -\frac{65}{84} & \frac{4}{7} & \frac{23}{84} & -\frac{1}{3} & \frac{1}{12} \\ -\frac{181}{588} & \frac{181}{147} & \frac{7}{12} & -\frac{24}{49} & -\frac{19}{84} & \frac{41}{147} & -\frac{41}{588} \\ \frac{23}{196} & -\frac{23}{49} & \frac{9}{28} & \frac{52}{49} & \frac{9}{28} & -\frac{23}{49} & \frac{23}{196} \\ -\frac{41}{588} & \frac{41}{147} & -\frac{19}{84} & -\frac{24}{49} & \frac{7}{12} & \frac{181}{147} & -\frac{181}{588} \\ \frac{1}{12} & -\frac{1}{3} & \frac{23}{84} & \frac{4}{7} & -\frac{65}{84} & -\frac{23}{21} & \frac{191}{84} \end{bmatrix} \begin{bmatrix} c^{k+1}_{2i-3} \\ c^{k+1}_{2i-2} \\ c^{k+1}_{2i-1} \\ c^{k+1}_{2i} \\ c^{k+1}_{2i+1} \\ c^{k+1}_{2i+2} \\ c^{k+1}_{2i+3} \end{bmatrix} \tag{3.2.2}$$

and the portion of this matrix expression that extracts an approximation for c^k_i is:

$$\begin{bmatrix} \frac{23}{196} & -\frac{23}{49} & \frac{9}{28} & \frac{52}{49} & \frac{9}{28} & -\frac{23}{49} & \frac{23}{196} \end{bmatrix} \begin{bmatrix} c^{k+1}_{2i-3} \\ c^{k+1}_{2i-2} \\ c^{k+1}_{2i-1} \\ c^{k+1}_{2i} \\ c^{k+1}_{2i+1} \\ c^{k+1}_{2i+2} \\ c^{k+1}_{2i+3} \end{bmatrix} \tag{3.2.3}$$

This corresponds precisely to (3.1.11).

A natural restriction to this least squares problem would be:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c^k_{i-1} \\ c^k_{i} \\ c^k_{i+1} \end{bmatrix} \approx \begin{bmatrix} c^{k+1}_{2i-1} \\ c^{k+1}_{2i} \\ c^{k+1}_{2i+1} \end{bmatrix} \tag{3.2.4}$$

which yields the solution:

$$\begin{bmatrix} -\frac{1}{2} & 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} c_{2i-1}^{k+1} \\ c_{2i}^{k+1} \\ c_{2i+1}^{k+1} \end{bmatrix} \tag{3.2.5}$$

This corresponds precisely to (3.1.10).

A natural extension would be:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_{i-3}^{k} \\ c_{i-2}^{k} \\ c_{i-1}^{k} \\ c_{i}^{k} \\ c_{i+1}^{k} \\ c_{i+2}^{k} \\ c_{i+3}^{k} \end{bmatrix} \approx \begin{bmatrix} c_{2i-5}^{k+1} \\ c_{2i-4}^{k+1} \\ c_{2i-3}^{k+1} \\ c_{2i-2}^{k+1} \\ c_{2i-1}^{k+1} \\ c_{2i}^{k+1} \\ c_{2i+1}^{k+1} \\ c_{2i+2}^{k+1} \\ c_{2i+3}^{k+1} \\ c_{2i+4}^{k+1} \\ c_{2i+5}^{k+1} \end{bmatrix} \tag{3.2.6}$$

which yields an 11-element vector of analysis coefficients:

$$\begin{array}{ccccccccccc} a_{i,2i-5}^{k+1} & a_{i,2i-4}^{k+1} & a_{i,2i-3}^{k+1} & a_{i,2i-2}^{k+1} & a_{i,2i-1}^{k+1} & a_{i,2i}^{k+1} & a_{i,2i+1}^{k+1} & a_{i,2i+2}^{k+1} & a_{i,2i+3}^{k+1} & a_{i,2i+4}^{k+1} & a_{i,2i+5}^{k+1} \\ \hline -\frac{569}{12038} & \frac{1138}{6019} & -\frac{141}{926} & -\frac{2024}{6019} & \frac{4479}{12038} & \frac{5714}{6019} & \frac{4479}{12038} & -\frac{2024}{6019} & -\frac{141}{926} & \frac{1138}{6019} & -\frac{569}{12038} \end{array} \tag{3.2.7}$$

Continuing expansions in this way, we would eventually end with a least squares approximation to c_i^k that includes all the fine points This is precisely the “global least squares” reversal of subdivision that we covered in [32]. Figures 6 and 7 compare the coefficients of (3.2.7) with global least squares analysis.

3.3 Equivalence

In Subsection 3.1 we solved underdetermined equation systems of the form

$$\mathbf{M}^T A = E \tag{3.3.1}$$

where E stands for a vector all of whose components are zero except for one, whose value is 1. Since the system is underdetermined, its general solution has the form

$$A = \mathbf{M}\alpha + \mathbf{Z}\beta \tag{3.3.2}$$

for some vectors of coefficients α and β, where the columns of $\mathbf{Z}$ form a basis for the nullspace of $\mathbf{M}^T$. In words, A is the sum of two vectors, one in the column space of $\mathbf{M}$ and one in the nullspace

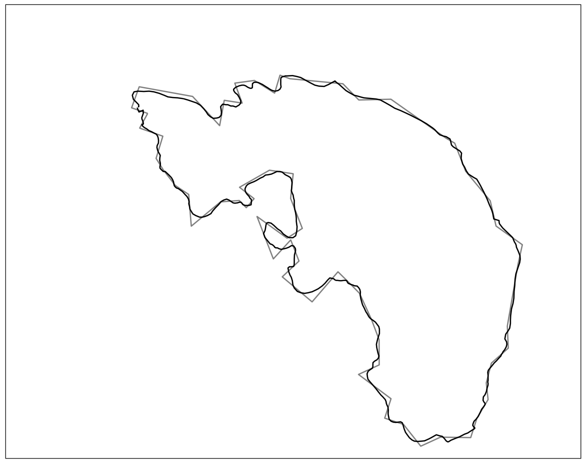


Figure 6: 11-element optimal analysis coefficients

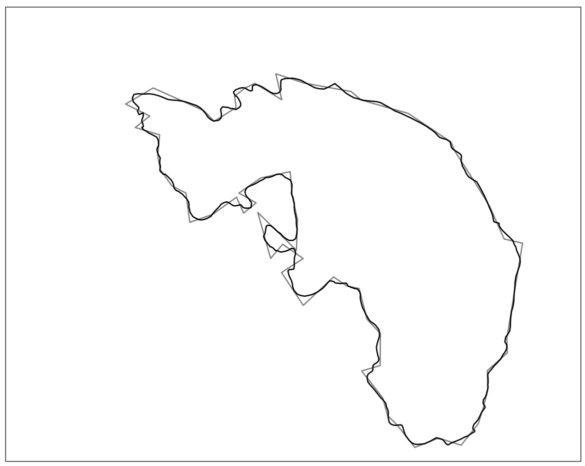


Figure 7: Global least squares analysis

of $\mathbf{M}^T$. The vector α is uniquely specified, and the totality of solutions to (3.3.1) is represented by varying the components of β over all real numbers. The square of the Euclidian norm of A is given by $\alpha^T\mathbf{M}^T\mathbf{M}\alpha + \beta^T\mathbf{Z}^T\mathbf{Z}\beta$, so the optimal solution is given by setting β to the zero vector. The result is then used as analysis coefficients:

$$A^T C^{k+1} = c_i^k \tag{3.3.3}$$

(Typically, the columns of $\mathbf{Z}$ are chosen to be orthonormal, so that $\beta^T\mathbf{Z}^T\mathbf{Z}\beta = \beta^T\beta$. Hence, near optimal solutions can be explored by looking at vectors β with small norm.)

In Subsection 3.2 we found least squares solutions to overdetermined equation systems of the form

$$\mathbf{M}C^k = C^{k+1} \tag{3.3.4}$$

where the matrices $\mathbf{M}$ were the same as those for (3.3.1). The solution has the form

$$C^k = \left(\mathbf{M}^T\mathbf{M}\right)^{-1}\mathbf{M}^T C^{k+1} \tag{3.3.5}$$

and we selected the component c_i^k of C^k, discarding the remaining components. This selection can be carried out by dotting C^k with a vector E that has zeros in all components except for a 1 in the position corresponding to c_i^k; namely, the same vector E that appears in (3.3.1). Thus, we have

$$
\begin{aligned}
E^T C^k = c_i^k &= E^T \left(\mathbf{M}^T\mathbf{M}\right)^{-1}\mathbf{M}^T C^{k+1} \\
&= A^T\mathbf{M}\left(\mathbf{M}^T\mathbf{M}\right)^{-1}\mathbf{M}^T C^{k+1} \\
&= \alpha^T\mathbf{M}^T\mathbf{M}\left(\mathbf{M}^T\mathbf{M}\right)^{-1}\mathbf{M}^T C^{k+1} \\
&= \alpha^T\mathbf{M}^T C^{k+1} \\
&= A^T C^{k+1}
\end{aligned}
\tag{3.3.6}
$$

exactly as in (3.3.3).

3.4 Finding the Coefficients $\mathbf{Q}^{k+1}$

We return to conditions (3.3). It is from these conditions that possible coefficients Q^{k+1} can be generated.

Take the near-optimal, 7-element version of A^{k+1} shown in Figure 4 as an example. The local pattern around $\lambda = i$ is:

λ $\downarrow$ \ ℓ $\rightarrow$	$2i-4$	$2i-3$	$2i-2$	$2i-1$	$2i$	$2i+1$	$2i+2$	$2i+3$	$2i+4$
$i-3$	$-\frac{1}{2}$	$\frac{1}{8}$							
$i-2$	1	$\frac{3}{8}$	$-\frac{1}{2}$	$\frac{1}{8}$					
$i-1$	$-\frac{1}{2}$	$\frac{3}{8}$	1	$\frac{3}{8}$	$-\frac{1}{2}$	$\frac{1}{8}$			
i		$\frac{1}{8}$	$-\frac{1}{2}$	$\frac{3}{8}$	1	$\frac{3}{8}$	$-\frac{1}{2}$	$\frac{1}{8}$	
$i+1$				$\frac{1}{8}$	$-\frac{1}{2}$	$\frac{3}{8}$	1	$\frac{3}{8}$	$-\frac{1}{2}$
$i+2$						$\frac{1}{8}$	$-\frac{1}{2}$	$\frac{3}{8}$	1
$i+3$								$\frac{1}{8}$	$-\frac{1}{2}$

(3.4.1)

This pattern extends in an obvious way, of course, to the left of $\ell = 2i-4$ for $\lambda = i-1, i-2, \ldots$, and it extends to the right of $\ell = 2i+4$ for $\lambda = i+1, i+2, \ldots$. However, for any κ for which $q_{\lambda,\kappa}^{k+1} = 0$ when $\lambda < i-3$ and $\lambda > i+3$, the remaining parts of the pattern would be unimportant. The subset $\mathbf{A}^{k+1}$ of the table that would remain important provides a nullspace problem to be solved:

$$\mathbf{A}^{k+1} Q^{k+1} = 0 \tag{3.4.2}$$

Thus, the first decision to make in constructing the elements of Q^{k+1} is the location of a convenient subset $\mathbf{A}^{k+1}$, which corresponds to the decision of what elements $q_{\lambda,\kappa}^{k+1}$ will be zero for each κ and what will be nonzero. In [32] we have given an extensive discussion of a decision methodology that will provide vectors orthogonal to patterns such as (3.4.1).

To paraphrase those discussions here, what we search for is the smallest subset of this pattern that will produce an underdetermined system of equations. Such a search proceeds incrementally as follows:

- If $q^{k+1}_{2i,\kappa}$ is nonzero, then we must include rows $i-1, i, i+1$ and column $2i$ in $\mathbf{A}^{k+1}$, which turns (3.4.2) into a 3×1 overdetermined system.

- If we allow $q^{k+1}_{2i-1,\kappa}$ to be nonzero as well, then $\mathbf{A}^{k+1}$ expands to include rows $i-2, i-1, i, i+1$ and columns $2i-1, 2i$, which turns (3.4.2) into a 4×2 overdetermined system.

- etc.

Proceeding by inspection, we find that the smallest underdetermined problem size is 6×7, an example of which is given by:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} q^{k+1}_{2i-4,\kappa} \\ q^{k+1}_{2i-3,\kappa} \\ q^{k+1}_{2i-2,\kappa} \\ q^{k+1}_{2i-1,\kappa} \\ q^{k+1}_{2i,\kappa} \\ q^{k+1}_{2i+1,\kappa} \\ q^{k+1}_{2i+2,\kappa} \end{bmatrix} = 0 \tag{3.4.3}$$

This system has the general solution:

$$\begin{array}{lcrl} q^{k+1}_{2i-4,\kappa} & = & & q^{k+1}_{2i+2,\kappa} \\ q^{k+1}_{2i-3,\kappa} & = & 4 & q^{k+1}_{2i+2,\kappa} \\ q^{k+1}_{2i-2,\kappa} & = & 3 & q^{k+1}_{2i+2,\kappa} \\ q^{k+1}_{2i-1,\kappa} & = & -8 & q^{k+1}_{2i+2,\kappa} \\ q^{k+1}_{2i,\kappa} & = & 3 & q^{k+1}_{2i+2,\kappa} \\ q^{k+1}_{2i+1,\kappa} & = & 4 & q^{k+1}_{2i+2,\kappa} \\ q^{k+1}_{2i+2,\kappa} & & & \text{(arbitrary)} \end{array} \tag{3.4.4}$$

and setting $q^{k+1}_{2i+2,\kappa}$ to $\frac{1}{8}$ produces

$$\begin{array}{ccccccc} q^{k+1}_{2i-4,\kappa} & q^{k+1}_{2i-3,\kappa} & q^{k+1}_{2i-2,\kappa} & q^{k+1}_{2i-1,\kappa} & q^{k+1}_{2i,\kappa} & q^{k+1}_{2i+1,\kappa} & q^{k+1}_{2i+2,\kappa} \\ \hline \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{array} \tag{3.4.5}$$

The general approach to finding a vector satisfying a nullspace equation such as (3.4.3) is as follows:

1. Reorder the columns of the matrix, if necessary, so that its leftmost columns form a nonsingular matrix. (If this cannot be done, the nullspace equation can be reduced at least by one row and column.)

2. The nullspace equation can be partitioned as follows:

$$\begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix} \begin{bmatrix} v_X \\ v_Y \end{bmatrix} = 0 \tag{3.4.6}$$

   where $\mathbf{X}$ is nonsingular. v_X is the portion of the (possibly reordered) nullspace vector corresponding to $\mathbf{X}$, and v_Y corresponds to $\mathbf{Y}$.

3. Choose v_Y arbitrarily (nonzero), and set

$$v_X = -\mathbf{X}^{-1}\mathbf{Y}v_Y \tag{3.4.7}$$

The verification that $[v_X\ v_Y]^T$ is a vector in the nullspace of the matrix is simple.

There is a short cut to this solution process if the section of the $\mathbf{A}$ matrix in question is two-slanted and its rows have identical nonzero elements; for example, the case of 5 nonzeros would be:

$$\begin{bmatrix} \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & d & e & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & b & c & d & e & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & a & b & c & d & e & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & a & b & c & d & e & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & a & b & c & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{3.4.8}$$

To find a matrix $\mathbf{Q}$ corresponding to this section; that is, one all of whose columns are in the nullspace of the section, we first fix on the number and location of positions in one of these columns where we will permit nonzeros to occur. One such position would interact with either two or three rows of $\mathbf{A}$, depending on where in the column that position is chosen to be. Two contiguous such column positions would interact with either two or three rows. And so on. Figure 8 shows how the interactions progress (indicated by arrows) for one selection sequence. By inspection it is evident that the addition of each two consecutive column positions will, on the average, result in only one further row interaction. At some point, the number of column positions will exceed the number of row interactions, and this will correspond to an underdetermined set of conditions. It will be further noticed that such an underdetermined situation in general settings will arise when the number of contiguous column positions equals the number of nonzeros in any row, and the placement of the contiguous positions is such that the rows interact in an even number of postions with the column.

In our example of the matrix section given in (3.4.8), the following is the first situation in which an

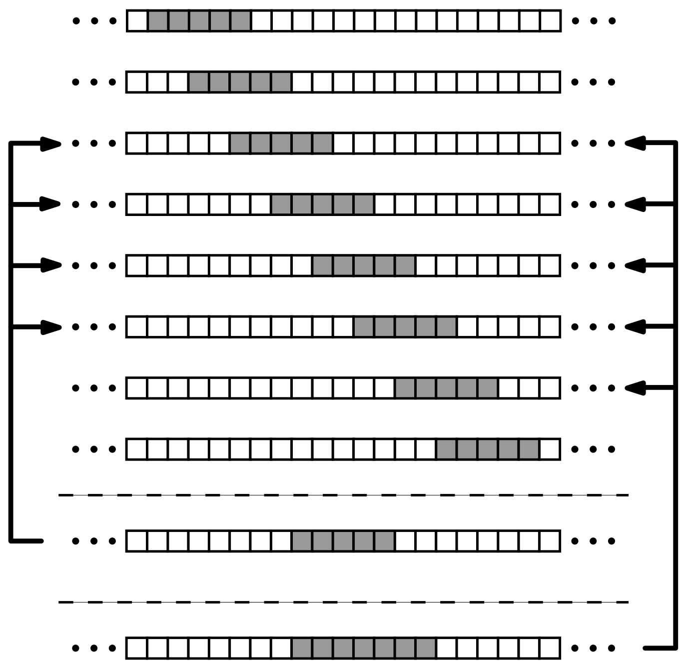


Figure 8: Interaction pattern example for a two-slanted system.

underdetermined set of conditions results:

$$\begin{array}{ccccccccccc} rd & + & se & & & & & & & = & 0 \\ rb & + & sc & + & td & + & ue & & & = & 0 \\ & & sa & + & tb & + & uc & + & vd & = & 0 \\ & & & & & & ua & + & vb & = & 0 \end{array} \tag{3.4.9}$$

where r, s, t, u, v represent the contents of the column positions in **Q**. The interaction of one of the rows of **A** and the column of **Q** in question is:

$$\begin{array}{lcccccccccc} \mathbf{A}: & \cdots & \cdots & 0 & a & b & c & d & e & 0 & \cdots \\ \mathbf{Q}: & \cdots & 0 & r & s & t & u & v & 0 & 0 & \cdots \end{array} \tag{3.4.10}$$

The third of the equations in (3.4.9) corresponds to the conditions that the two vectors in (3.4.10) have a zero dot product. A classical trick to achieve this zero dot product, verifiable by inspection, is to let the nonzeros in the vector **Q** be the nonzeros in the vector **A**, reversed in order and alternating in sign:

$$\begin{array}{lcccccccccc} \mathbf{A}: & \cdots & \cdots & 0 & a & b & c & d & e & 0 & \cdots \\ \mathbf{Q}: & \cdots & 0 & e & -d & c & -b & a & 0 & 0 & \cdots \end{array} \tag{3.4.11}$$

The other equations of (3.4.9) will also be satisfied by this choice, in this example as well as in the general two-slanted situation.

These observations lead to a short cut for the general, shifted column of $\mathbf{Q}$, corresponding to a nullspace equation of the form (3.4.8), of whatever size, provided $\mathbf{A}$ is two-slanted. The solution to the nullspace equation is given by a column of $\mathbf{Q}$ that contains the nonzeros of the general column of $\mathbf{A}$, reversed in order, alternated in sign, and shifted in row position so as to overlap the nonzeros of $\mathbf{A}$ in an even number of rows. For the specific case given in (3.4.3) for example, we take the vector,

$$\begin{bmatrix} \cdots & 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & \cdots \end{bmatrix} \tag{3.4.12}$$

shift the entries so that the nonzero vector entries overlap the row entries in an even number of positions,

$$\begin{bmatrix} \cdots & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{3.4.13}$$

reverse the order of the nonzero entries (which is invisible in this case because of symmetry),

$$\begin{bmatrix} \cdots & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{3.4.14}$$

and multiply each alternate entry by -1

$$\begin{bmatrix} \cdots & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{3.4.15}$$

and we arrive at the solution given in (3.4.5). For situations that are not two-slanted, of course, a solution can be found by linear algebra; e.g. as indicated in (3.4.7). However, since a vast number of univariate subdivisions have two-slanted matrices, this short cut is worth mentioning.

The short cut we have described in the finite case is something that is well known in the infinite setting of wavelets; for example, see the reference by Stollnitz, et. al. [34]. To quote from this reference: “This recipe for creating a wavelet sequence from a scaling function sequence [sequence reversal together with sign alternation] is common to many wavelet constructions on the infinite real line; such sequences are known as *quadrature mirror filters*.”

How we assign the index κ is a matter of our convenience (provided, ultimately, that the indices of the coefficients in the sets P^{k+1}, Q^{k+1}, A^{k+1}, and B^{k+1} are in conformance). So we shall let $\kappa = i$ in (3.4.5). Replacing i by $i+j$ for $j = \ldots, -1, 0, 1, \ldots$ generates the remaining parts of Q^{k+1} associated with interior points of the data. For the boundary points, special systems of the form (3.4.2) must be constructed and solved; [32] contains a discussion.

3.5 Finding the Coefficients $\mathbf{B}^{k+1}$

In order for (1.4) and (1.2) to be consistent, we must have:

$$\begin{aligned} d^k_\kappa &= \sum_{\ell \in \dot{\mathcal{B}}^{k+1}_\kappa} b^{k+1}_{\kappa,\ell} c^{k+1}_\ell \\ &= \sum_{\mu \in \ddot{\mathcal{P}}^{k+1}_\ell} \left(\sum_{\ell \in \dot{\mathcal{B}}^{k+1}_\kappa} b^{k+1}_{\kappa,\ell} p^{k+1}_{\ell,\mu} \right) c^k_\mu + \sum_{\nu \in \ddot{\mathcal{Q}}^{k+1}_\ell} \left(\sum_{\ell \in \dot{\mathcal{B}}^{k+1}_\kappa} b^{k+1}_{\kappa,\ell} q^{k+1}_{\ell,\nu} \right) d^k_\nu \end{aligned} \tag{3.5.1}$$

implying

$$\sum_{\ell \in \dot{\mathcal{B}}^{k+1}_\kappa} b^{k+1}_{\kappa,\ell} q^{k+1}_{\ell,\nu} = \delta_{\kappa,\nu} \tag{3.5.2}$$

and

$$\sum_{\ell \in \dot{\mathcal{B}}_{\kappa}^{k+1}} b_{\kappa,\ell}^{k+1} p_{\ell,\mu}^{k+1} = 0 \tag{3.5.3}$$

(Note that (3.5.2) and (3.5.3) represent the bottom portion of (2.5).)

The final phase of the construction process, producing B^{k+1}, echoes the construction of A^{k+1}. Now however, the elements of Q^{k+1} play the role formerly played by P^{k+1}. We shall construct B^{k+1} to satisfy (3.5.2). The expectation that (3.5.3) also holds will be covered in Section 5.

Take equation (1.2) and ignore the first summation to obtain:

$$c_{\ell}^{k+1} = \sum_{\kappa \in \ddot{\mathcal{Q}}_{\ell}^{k+1}} q_{\ell,\kappa}^{k+1} d_{\kappa}^{k} \tag{3.5.4}$$

Focusing on c_{2i}^{k+1} yields:

$$\begin{aligned}
c_{2i-5}^{k+1} &= \tfrac{1}{2}d_{i-3}^{k} - d_{i-2}^{k} + \tfrac{1}{2}d_{i-1}^{k} \\
c_{2i-4}^{k+1} &= \tfrac{1}{8}d_{i-3}^{k} + \tfrac{3}{8}d_{i-2}^{k} + \tfrac{3}{8}d_{i-1}^{k} + \tfrac{1}{8}d_{i}^{k} \\
c_{2i-3}^{k+1} &= \tfrac{1}{2}d_{i-2}^{k} - d_{i-1}^{k} + \tfrac{1}{2}d_{i}^{k} \\
c_{2i-2}^{k+1} &= \tfrac{1}{8}d_{i-2}^{k} + \tfrac{3}{8}d_{i-1}^{k} + \tfrac{3}{8}d_{i}^{k} + \tfrac{1}{8}d_{i+1}^{k} \\
c_{2i-1}^{k+1} &= \tfrac{1}{2}d_{i-1}^{k} - d_{i}^{k} + \tfrac{1}{2}d_{i+1}^{k} \\
c_{2i}^{k+1} &= \tfrac{1}{8}d_{i-1}^{k} + \tfrac{3}{8}d_{i}^{k} + \tfrac{3}{8}d_{i+1}^{k} + \tfrac{1}{8}d_{i+2}^{k} \\
c_{2i+1}^{k+1} &= \tfrac{1}{2}d_{i}^{k} - d_{i+1}^{k} + \tfrac{1}{2}d_{i+2}^{k} \\
c_{2i+2}^{k+1} &= \tfrac{1}{8}d_{i}^{k} + \tfrac{3}{8}d_{i+1}^{k} + \tfrac{3}{8}d_{i+2}^{k} + \tfrac{1}{8}d_{i+3}^{k} \\
c_{2i+3}^{k+1} &= \tfrac{1}{2}d_{i+1}^{k} - d_{i+2}^{k} + \tfrac{1}{2}d_{i+3}^{k}
\end{aligned} \tag{3.5.5}$$

Multiply the equation in c_{ℓ}^{k+1} by $b_{i,\ell}^{k+1}$ and add up. The implications to be drawn from the result yield a matrix equation corresponding to (3.1.8):

$$\begin{bmatrix}
\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\
0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 \\
0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
b_{i,2i-5}^{k+1} \\ b_{i,2i-4}^{k+1} \\ b_{i,2i-3}^{k+1} \\ b_{i,2i-2}^{k+1} \\ b_{i,2i-1}^{k+1} \\ b_{i,2i}^{k+1} \\ b_{i,2i+1}^{k+1} \\ b_{i,2i+2}^{k+1} \\ b_{i,2i+3}^{k+1}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0
\end{bmatrix} \tag{3.5.6}$$

This matrix equation is not sufficient to enforce geometric validity, however. In order to respect equation

(2.4), the matrix equation must be expanded to one resembling (3.1.7):

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 & \frac{3}{8} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} b_{i,2i-5}^{k+1} \\ b_{i,2i-4}^{k+1} \\ b_{i,2i-3}^{k+1} \\ b_{i,2i-2}^{k+1} \\ b_{i,2i-1}^{k+1} \\ b_{i,2i}^{k+1} \\ b_{i,2i+1}^{k+1} \\ b_{i,2i+2}^{k+1} \\ b_{i,2i+3}^{k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.5.7}$$

This matrix equation has the following general solution:

$$\begin{array}{rcrl} b_{i,2i-5}^{k+1} & = & - & b_{i,2i+3}^{k+1} \\ b_{i,2i-4}^{k+1} & = & 4 & b_{i,2i+3}^{k+1} \\ b_{i,2i-3}^{k+1} & = & -2 & b_{i,2i+3}^{k+1} - \frac{1}{8} \\ b_{i,2i-2}^{k+1} & = & -12 & b_{i,2i+3}^{k+1} + \frac{1}{2} \\ b_{i,2i-1}^{k+1} & = & -\frac{3}{4} & \\ b_{i,2i}^{k+1} & = & 12 & b_{i,2i+3}^{k+1} + \frac{1}{2} \\ b_{i,2i+1}^{k+1} & = & 2 & b_{i,2i+3}^{k+1} - \frac{1}{8} \\ b_{i,2i+2}^{k+1} & = & -4 & b_{i,2i+3}^{k+1} \\ b_{i,2i+3}^{k+1} & = & & (\text{arbitrary}) \end{array} \tag{3.5.8}$$

Clearly the minimal norm solution is given when $b_{2i+3} = 0$:

$$\begin{array}{ccccccccc} b_{i,2i-5}^{k+1} & b_{i,2i-4}^{k+1} & b_{i,2i-3}^{k+1} & b_{i,2i-2}^{k+1} & b_{i,2i-1}^{k+1} & b_{i,2i}^{k+1} & b_{i,2i+1}^{k+1} & b_{i,2i+2}^{k+1} & b_{i,2i+3}^{k+1} \\ \hline 0 & 0 & -\frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & \frac{1}{2} & -\frac{1}{8} & 0 & 0 \end{array} \tag{3.5.9}$$

The results of Subsection 3.3 hold here as well. This minimal solution is also given by the 5^{th} row of $\left(\mathbf{M}\mathbf{M}^T\right)^{-1}\mathbf{M}$, where $\mathbf{M}$ is the matrix in (3.5.7).

4 Matrices and Inner Products

Except for the interpretation of a biorthogonal system as a semiorthogonal system having a different inner product, the material in this section reflects and summarizes material in [34].

We have taken pains to express our problems and the results in terms of the local indices i and $2i$ and the individual coefficients. Here, however, we shall show small examples of the results in a matrix format. (1.2) is often given as the matrix equation

$$C^{k+1} = \mathbf{P}^{k+1} C^k + \mathbf{Q}^{k+1} D^k \tag{4.1}$$

and for the case in which the C^{k+1} would consist of 10 points on a closed curve, $\mathbf{P}^{k+1}$ and $\mathbf{Q}^{k+1}$ would be as follows:

$$\mathbf{P}^{k+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{3}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{8} & 0 & 0 & \frac{1}{8} \end{bmatrix} \tag{4.2}$$

$$\mathbf{Q}^{k+1} = \begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & \frac{1}{8} \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{8} & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -1 \\ \frac{3}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{3}{8} \end{bmatrix} \tag{4.3}$$

Similarly (1.3) and (1.4) have the matrix form:

$$\begin{aligned} C^k &= \mathbf{A}^{k+1} C^{k+1} \\ D^k &= \mathbf{B}^{k+1} C^{k+1} \end{aligned} \tag{4.4}$$

for which we have:

$$\mathbf{A}^{k+1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 \\ \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} \\ \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 \\ \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} \end{bmatrix} \tag{4.5}$$

$$\mathbf{B}^{k+1} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & \frac{1}{2} & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & \frac{1}{2} & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & \frac{1}{2} & -\frac{1}{8} & 0 \\ -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & \frac{1}{2} \end{bmatrix} \tag{4.6}$$

Returning to (1.7) and (1.8), we see that

$$\begin{aligned} \delta_{m,\lambda} &= \left\langle \tilde{\phi}_m^k, \phi_\lambda^k \right\rangle \\ &= \left\langle \sum_{i \in \ddot{\tilde{\mathcal{A}}}_m^{k+1}} \tilde{a}_{m,i}^{k+1} \phi_i^{k+1}, \sum_{j \in \dot{\mathcal{P}}_\lambda^{k+1}} p_{j,\lambda}^{k+1} \phi_j^{k+1} \right\rangle \\ &= \sum_{i \in \ddot{\tilde{\mathcal{A}}}_m^{k+1}} \sum_{j \in \dot{\mathcal{P}}_\lambda^{k+1}} \tilde{a}_{m,i}^{k+1} \left\langle \phi_i^{k+1}, \phi_j^{k+1} \right\rangle p_{j,\lambda}^{k+1} \end{aligned} \tag{4.7}$$

That is,

$$\mathbf{I} = \tilde{\mathbf{A}}^{k+1} \mho^{k+1} \mathbf{P}^{k+1} \tag{4.8}$$

where

$$\mho^{k+1} = \begin{bmatrix} & \vdots & \\ \cdots & \left\langle \phi_i^{k+1}, \phi_j^{k+1} \right\rangle & \cdots \\ & \vdots & \end{bmatrix} \tag{4.9}$$

is the *Gram matrix* for the inner product on the space $\mathcal{V}^{k+1}$. Likewise

$$\begin{aligned} \mathbf{I} &= \tilde{\mathbf{B}}^{k+1} \mho^{k+1} \mathbf{Q}^{k+1} \\ \mathbf{0} &= \tilde{\mathbf{A}}^{k+1} \mho^{k+1} \mathbf{Q}^{k+1} \\ \mathbf{0} &= \tilde{\mathbf{B}}^{k+1} \mho^{k+1} \mathbf{P}^{k+1} \end{aligned} \tag{4.10}$$

This provides us with the identities

$$\begin{aligned} \mathbf{A}^{k+1} &= \tilde{\mathbf{A}}^{k+1} \mho^{k+1} \\ \mathbf{B}^{k+1} &= \tilde{\mathbf{B}}^{k+1} \mho^{k+1} \end{aligned} \tag{4.11}$$

Moreover, C^k and D^k are the projections of C^{k+1} onto their respective subspaces, range$(\mathbf{P}^{k+1})$ and range$(\mathbf{Q}^{k+1})$:

$$\begin{aligned} C^k &= \left(\mathbf{P}^{k+1^T}\mho^{k+1}\mathbf{P}^{k+1}\right)^{-1}\mathbf{P}^{k+1^T}\mho^{k+1}C^{k+1} \\ &= \left(\mathbf{P}^{k+1^T}\mho^{k+1}\mathbf{P}^{k+1}\right)^{-1}\mathbf{P}^{k+1^T}\mho^{k+1}\left[\mathbf{P}^{k+1}C^k + \mathbf{Q}^{k+1}D^k\right] \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} D^k &= \left(\mathbf{Q}^{k+1^T}\mho^{k+1}\mathbf{Q}^{k+1}\right)^{-1}\mathbf{Q}^{k+1^T}\mho^{k+1}C^{k+1} \\ &= \left(\mathbf{Q}^{k+1^T}\mho^{k+1}\mathbf{Q}^{k+1}\right)^{-1}\mathbf{Q}^{k+1^T}\mho^{k+1}\left[\mathbf{P}^{k+1}C^k + \mathbf{Q}^{k+1}D^k\right] \end{aligned} \tag{4.13}$$

which provides us with two more identities:

$$\begin{aligned} \mathbf{A}^{k+1} &= \tilde{\mathbf{A}}^{k+1}\mho^{k+1} &= \left(\mathbf{P}^{k+1^T}\mho^{k+1}\mathbf{P}^{k+1}\right)^{-1}\mathbf{P}^{k+1^T}\mho^{k+1} \\ \mathbf{B}^{k+1} &= \tilde{\mathbf{B}}^{k+1}\mho^{k+1} &= \left(\mathbf{Q}^{k+1^T}\mho^{k+1}\mathbf{Q}^{k+1}\right)^{-1}\mathbf{Q}^{k+1^T}\mho^{k+1} \end{aligned} \tag{4.14}$$

A multiresolution system is semiorthogonal when

$$\mathbf{P}^{k+1^T}\mho^{k+1}\mathbf{Q}^{k+1} = \mathbf{0} \tag{4.15}$$

In that context, we note that any biorthogonal system is semiorthogonal with respect to some inner product:

$$\mathbf{P}^{k+1^T}\left[\mathbf{A}^{k+1^T}\ \mathbf{B}^{k+1^T}\right]\left[\begin{array}{c}\mathbf{A}^{k+1}\\ \mathbf{B}^{k+1}\end{array}\right]\mathbf{Q}^{k+1} = \mathbf{0} \tag{4.16}$$

where:

$$\left[\mathbf{A}^{k+1^T}\ \mathbf{B}^{k+1^T}\right]\left[\begin{array}{c}\mathbf{A}^{k+1}\\ \mathbf{B}^{k+1}\end{array}\right] = \mho^{k+1} \tag{4.17}$$

Plugging (4.17) into (4.12) and (4.13) bears this out.

Using this, the Gram matrix for our construction for cubic B-spline subdivision is:

$$\mho^{k+1} = \begin{bmatrix} \frac{29}{32} & -\frac{5}{16} & \frac{27}{64} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{27}{64} & -\frac{5}{16} \\ -\frac{5}{16} & 2 & -\frac{5}{16} & -\frac{3}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & -\frac{3}{4} \\ \frac{27}{64} & -\frac{5}{16} & \frac{29}{32} & -\frac{5}{16} & \frac{27}{64} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{3}{4} & -\frac{5}{16} & 2 & -\frac{5}{16} & -\frac{3}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} & \frac{27}{64} & -\frac{5}{16} & \frac{29}{32} & -\frac{5}{16} & \frac{27}{64} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & -\frac{3}{4} & -\frac{5}{16} & 2 & -\frac{5}{16} & -\frac{3}{4} & -\frac{1}{8} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{27}{64} & -\frac{5}{16} & \frac{29}{32} & -\frac{5}{16} & \frac{27}{64} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & -\frac{3}{4} & -\frac{5}{16} & 2 & -\frac{5}{16} & -\frac{3}{4} \\ \frac{27}{64} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{27}{64} & -\frac{5}{16} & \frac{29}{32} & -\frac{5}{16} \\ -\frac{5}{16} & -\frac{3}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & -\frac{3}{4} & -\frac{5}{16} & 2 \end{bmatrix} \tag{4.18}$$

The inner product with respect to which we are establishing a multiresolution system can be inferred from $\mho^{k+1}$. Given any $f,g \in \mathcal{V}^{k+1}$, represent f as $f = \sum_i f_i \phi_i^{k+1}$, g as $g = \sum_j g_j \phi_j^{k+1}$, then $\langle f,g \rangle = F^T \mho^{k+1} G$, where F and G represent the coefficient vectors in the representations of f and g respectively.

If we had known about the inner product represented by $\mho^{k+1}$ in advance and had sought to find a semiorthogonal system with respect to both the inner product and the subdivision represented by $\mathbf{P}^{k+1}$, then we would have formed the product:

$$\mathbf{P}^{k+1^T} \mho^{k+1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 \\ \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} \\ \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{8} & 0 \\ \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} & \frac{3}{8} & 1 & \frac{3}{8} & -\frac{1}{2} \end{bmatrix} \tag{4.19}$$

and used the shortcut from [32] given in (3.4.12) through (3.4.15) to find a matrix $\mathbf{Q}^{k+1^T}$ orthogonal to this product. We would have obtained the same results as those given in (3.4.5) and repeated in (4.3).

5 General Construction

The elements of the construction have been as follows:

1. Given a subdivision, select any representative fine point c_μ^{k+1}, as shown in (1.1).

2. Decide which coarse point c_γ^k is to be associated with c_μ^{k+1}, and write down all equations involving this coarse point. These equations will involve fine points in some connection neighborhood of c_μ^{k+1}.

3. Optionally, add additional equations involving fine points in some enclosing connection neighborhood of c_μ^{k+1}.

4. The selected equations can be written as a matrix system corresponding to (3.2.1). The object of the selection process is to provide such a matrix system that is *overdetermined*. If this selection was made correctly, each row of the matrix will sum to one.

5. Alternatively, the matrix can be transposed (as (3.3.1)) to provide an *underdetermined* system for filter coefficients A^{k+1} local to $a_{\gamma,\mu}^{k+1}$, yielding a matrix system corresponding to (3.1.1).

6. The optimal solution to the system in step 5 is given by solving the overdetermined system of step 4 in the least squares sense. Nearby solutions can be explored by finding the general solution (as in (3.3.2)) of the system in step 5.

7. Having produced the elements of A^{k+1}, focus on the elements local to $a_{\gamma,\mu}^{k+1}$; that is, those $a_{\lambda,\ell}^{k+1}$ whose first index is in the index neighborhood of γ and those whose second index is in the index neighborhood of μ. (The array (3.4.1) shows an example with $\gamma = i$ and $\mu = 2i$.) Search for a set of interactions between unknown $q_{\ell,\kappa}^{k+1}$ that represent an underdetermined equation set. Figure 8 illustrates how this search proceeds.

8. Using the information from step 7 and the steps laid out in subsection 3.4, produce elements of Q^{k+1} that interact with $a^{k+1}_{\gamma,\mu}$ and its neighboring elements of A^{k+1}.

9. The elements of Q^{k+1} contribute to the definition of c^{k+1}_{μ} according to the reconstruction equations (1.2). As in step 2, focus on the coarse point c^{k}_{γ} and its corresponding fine point c^{k+1}_{μ}. Select the Q portion of equations (1.2) for this index pair as well as adjacent first indices about γ and second indices about μ as in (3.5.5).

10. From the equations of step 9 form a matrix system for elements of B^{k+1} according to the model of (3.5.6), and augment this system as shown in (3.5.7). The object of step 9 is to choose equations so that the result is in underdetermined system for the elements of B^{k+1} local to the index pair (γ,μ).

11. Finish the process by solving the system in step 10. As in step 6, the transpose of the system in step 10 will yield the optimal solution via least squares, and other solutions may be explored directly from the underdetermined system of step 10.

Steps 1 through 11 must be carried out for each distinctly different connection neighborhood of the subdivision; typically, this will be once for each generic category of interior point and once for each generic category of boundary point.

We are not in a position to give any theory establishing when this construction process can be expected to work. However, the intuition and observations we used to invent the construction may serve as insights to others who could provide necessary and/or sufficient conditions for its success. The intuition and observations might also lend a feeling of hope to those who want to experiment with the construction.

Each step of the construction depends upon finding an underdetermined set of equations to solve. Except for boundary situations, this is accomplished by taking a regular slice of a given matrix and exploring how the rows or columns of that matrix interact with potentially nonzero locations in the general column or row of a matrix to be constructed. If the regular slice is two-slanted or better, then Figure 8 illustrates how such an exploration proceeds and indicates that it must arrive at an underdetermined system involving a small portion of the regular slice.

The solution of the underdetermined solution must succeed, if the regular slice portion has full rank. This is expected to be true for slanted systems directly from the shifted structure of their rows or columns. By the regularity of the situation, the solution produces one row or column of a constructed matrix that echos the slanting of the given matrix.

At the end of the full process, the vectors of coefficients in A^{k+1} will have been constructed to span the range space of the matrix formed by the coefficients P^{k+1}. The coefficients Q^{k+1} have then been constructed to span the nullspace of the matrix formed by the coefficients A^{k+1}, which makes them span the nullspace of P^{k+1}. The construction to follow will generate vectors of coefficients B^{k+1} that span the range space of the matrix formed by the coefficients Q^{k+1}, which will put them in the nullspace of P^{k+1}, causing (3.5.3) to hold as an additional result.

6 Two Further Examples

To gain more experience with this approach, we offer two further examples. In subsection 6.1, we carry out our approach on Chaikin (quadratic B-spline) subdivision. The optimal A^{k+1} coefficients comprise simply the numbers $\frac{1}{4}$ and $\frac{3}{4}$, as do all other filter coefficients, which produces a particularly simple and appealing system. In Subsection 6.2, to depart from B-splines, we build a system of filter coefficients for the curve subdivision due to Dyn, Levin, and Gregory [17]. This also provides an example of our approach applied to an interpolatory subdivision.

6.1 A Simpler B-Spline System: Chaikin

We provide another example of this approach for which the results were particularly good, namely the example provided by Chaikin's curve subdivision, for which the underlying scale functions are the quadratic B-splines.

Chaikin subdivision is given by:

$$\begin{aligned} c_{2i}^{k+1} &= p_{2i,i-1}^{k+1} c_{i-1}^{k} + p_{2i,i}^{k+1} c_i^{k+1} = \tfrac{1}{4} c_{i-1}^{k} + \tfrac{3}{4} c_i^{k} \\ c_{2i+1}^{k+1} &= p_{2i+1,i}^{k+1} c_i^{k} + p_{2i+1,i+1}^{k+1} c_{i+1}^{k+1} = \tfrac{3}{4} c_i^{k} + \tfrac{1}{4} c_{i+1}^{k} \end{aligned} \tag{6.1.1}$$

An optimal set of A^{k+1} coefficients of length 4 is given by:

$$\begin{aligned} c_i^k &= a_{i,2i-1}^{k+1} c_{2i-1}^{k+1} + a_{i,2i}^{k+1} c_{2i}^{k+1} + a_{i,2i+1}^{k+1} c_{2i+1}^{k+1} + a_{i,2i+2}^{k+1} c_{2i+2}^{k+1} \\ &= -\tfrac{1}{4} c_{2i-1}^{k+1} + \tfrac{3}{4} c_{2i}^{k+1} + \tfrac{3}{4} c_{2i+1}^{k+1} - \tfrac{1}{4} c_{2i+2}^{k+1} \end{aligned} \tag{6.1.2}$$

The corresponding Q^{k+1} coefficients appear in the following:

$$\begin{aligned} c_{2i}^{k+1} &= \mathbf{P}^{k+1} C^k + q_{2i,i-1}^{k+1} d_{i-1}^{k} + q_{2i,i}^{k+1} d_i^{k+1} \\ &= \mathbf{P}^{k+1} C^k + \tfrac{1}{4} d_{i-1}^{k} - \tfrac{3}{4} d_i^{k} \\ c_{2i+1}^{k+1} &= \mathbf{P}^{k+1} C^k + q_{2i+1,i}^{k+1} d_i^{k} + q_{2i+1,i+1}^{k+1} d_{i+1}^{k+1} \\ &= \mathbf{P}^{k+1} C^k + \tfrac{3}{4} d_i^{k} - \tfrac{1}{4} d_{i+1}^{k} \end{aligned} \tag{6.1.3}$$

where $\mathbf{P}^{k+1} C^k$ hides the terms in (6.1.1). Finally, the B^{k+1} coefficients are given by:

$$\begin{aligned} d_i^k &= b_{i,2i-1}^{k+1} c_{2i-1}^{k+1} + b_{i,2i}^{k+1} c_{2i}^{k+1} + b_{i,2i+1}^{k+1} c_{2i+1}^{k+1} + b_{i,2i+2} c_{2i+2}^{k+1} \\ &= \tfrac{1}{4} c_{2i-1}^{k+1} - \tfrac{3}{4} c_{2i}^{k+1} + \tfrac{3}{4} c_{2i+1}^{k+1} - \tfrac{1}{4} c_{2i+2}^{k+1} \end{aligned} \tag{6.1.4}$$

In matrix terms for a small cyclic system this would amount to:

$$\mathbf{P}^{k+1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix} \tag{6.1.5}$$

$$\mathbf{Q}^{k+1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ \frac{1}{4} & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & 0 & 0 & \frac{3}{4} \\ -\frac{3}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix} \tag{6.1.6}$$

$$\mathbf{A}^{k+1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \tag{6.1.7}$$

$$\mathbf{B}^{k+1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \tag{6.1.8}$$

A small section of a Gram matrix induced by this system of coefficients would be:

$$\mho^{k+1} = \begin{bmatrix} \frac{5}{4} & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{4} & -\frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & \frac{5}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{4} & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{4} & \frac{5}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & \frac{5}{4} \end{bmatrix} \tag{6.1.9}$$

6.2 4-Point Interpolatory Subdivision

The version of the subdivision given in [17] that we use will be with $w = \frac{1}{16}$:

$$\begin{aligned} c_{2i}^{k+1} &= p_{2i,i}^{k+1} c_i^{k+1} \\ &= 1\, c_i^k \\ c_{2i+1}^{k+1} &= p_{2i+1,i-1}^{k+1} c_{i-1}^k + p_{2i+1,i}^{k+1} c_i^k + p_{2i+1,i+1}^{k+1} c_{i+1}^{k+1} + p_{2i+1,i+2}^{k+1} c_{i+2}^{k+1} \\ &= -\tfrac{1}{16} c_{i-1}^k + \tfrac{9}{16} c_i^k + \tfrac{9}{16} c_{i+1}^k - \tfrac{1}{16} c_{i+2}^k \end{aligned} \tag{6.2.1}$$

Clearly the association of each coarse point c_i^k is with the fine point c_{2i}^{k+1}. Just as clearly there is an obvious multiresolution to be formed: decimate the $k+1$-points by taking those of even index as the k-points and retain the residuals $c_{2i+1}^{k+1} - \left(-\frac{1}{16}c_{i-1}^k + \frac{9}{16}c_i^k + \frac{9}{16}c_{i+1}^k - \frac{1}{16}c_{i+2}^k\right)$ as the detail information (required only for the fine points of odd index). Hence, $2N$ items of fine data would be represented as N items of coarse data and N items of detail information, which is what one generally expects from one stage of a multiresolution. This is suggestive of Faber's treatment of piecewise linear data.

We reject this approach for two reasons. Firstly, it is too obvious and too trivial to represent any contribution. Secondly, going back to the original motivation for considering the reversal of subdivision rules, we would like to assume that the data C^{k+1} might come from measurements; e.g., from a laser range finder. As such, any point c_{2i}^{k+1} would be associated with some measurement error, and we might hope that a coarse point c_i^k that was *approximated* from c_{2i}^{k+1} and its surrounding points would be a better choice than would be taking c_i^k as any one of the fine points alone.

An optimal set of A^{k+1} coefficients of length 9 is given by:

$$\begin{aligned} c_i^k &= \textstyle\sum_{j=-4}^{j=4} a_{i,2i+j}^{k+1}\, c_{2i+j} \\ &= \tfrac{3}{161}\, c_{2i-4}^{k+1} + 0\, c_{2i-3}^{k+1} - \tfrac{24}{161}\, c_{2i-2}^{k+1} \\ &+ \tfrac{48}{161}\, c_{2i-1}^{k+1} + \tfrac{107}{161}\, c_{2i}^{k+1} + \tfrac{48}{161}\, c_{2i+1}^{k+1} \\ &- \tfrac{24}{161}\, c_{2i+2}^{k+1} + 0\, c_{2i+3}^{k+1} + \tfrac{3}{161}\, c_{2i+4}^{k+1} \end{aligned} \tag{6.2.2}$$

The norm of this set is 0.6645962733. Interestingly, any optimal set of A^{k+1} coefficients of length less than 9 reduces simply to $a_{i,2i}^{k+1} = 1$ and $a_{i,j}^{k+1} = 0$ for $j \neq 2i$, which corresponds to the trivial (Faber-style) decimation that we rejected at the beginning of this section.

A suboptimal set of A^{k+1} coefficients of length 9 is given by:

$$\begin{aligned} c_i^k &= \tfrac{1}{64}\, c_{2i-4}^{k+1} + 0\, c_{2i-3}^{k+1} - \tfrac{1}{8}\, c_{2i-2}^{k+1} \\ &+ \tfrac{1}{4}\, c_{2i-1}^{k+1} + \tfrac{23}{32}\, c_{2i}^{k+1} + \tfrac{1}{4}\, c_{2i+1}^{k+1} \\ &- \tfrac{1}{8}\, c_{2i+2}^{k+1} + 0\, c_{2i+3}^{k+1} + \tfrac{1}{64}\, c_{2i+4}^{k+1} \end{aligned} \tag{6.2.3}$$

This set has the advantage that it comprises only inverse powers of 2. Moreover, the norm of this set is 0.6733398438, which is only slightly larger than the optimal set of the same length. It is this set of A^{k+1} that we use to generate the remaining filter coefficients.

The corresponding Q^{k+1} coefficients are given by:

$$
\begin{aligned}
c_{2i}^{k+1} &= \mathbf{P}^{k+1}C^k &&+ q_{2i,i}^{k+1}\, d_i^k &&+ q_{2i,i+1}^{k+1}\, d_{i+1}^{k+1} \\
&= \mathbf{P}^{k+1}C^k &&- \tfrac{1}{4}\, d_i^k &&- \tfrac{1}{4}\, d_{i+1}^k \\
c_{2i+1}^{k+1} &= \mathbf{P}^{k+1}C^k &&+ q_{2i+1,i-1}^{k+1}\, d_{i-1}^k &&+ q_{2i+1,i}^{k+1}\, d_i^{k+1} \\
&+ q_{2i+1,i+1}^{k+1}\, d_{i+1}^k &&+ q_{2i+1,i+2}^{k+1}\, d_{i+2}^{k+1} &&+ q_{2i+1,i+3}^{k+1}\, d_{i+3}^{k+1} \\
&= \mathbf{P}^{k+1}C^k &&+ \tfrac{1}{64}\, d_{i-1}^k &&- \tfrac{1}{8}\, d_i^{k+1} \\
&+ \tfrac{23}{32}\, d_{i+1}^k &&- \tfrac{1}{8}\, d_{i+2}^{k+1} &&+ \tfrac{1}{64}\, d_{i+3}^{k+1}
\end{aligned}
\tag{6.2.4}
$$

where $\mathbf{P}^{k+1}C^k$ hides the terms in (6.2.1).

Finally, the B^{k+1} coefficients are given by:

$$
\begin{aligned}
d_i^k &= b_{i,2i-4}^{k+1}\, c_{2i-4}^{k+1} &&+ b_{i,2i-3}^{k+1}\, c_{2i-3}^{k+1} &&+ b_{i,2i-2}^{k+1}\, c_{2i-2}^{k+1} &&+ b_{i,2i-1}^{k+1}\, c_{2i-1}^{k+1} \\
&+ b_{i,2i}^{k+1}\, c_{2i}^{k+1} &&+ b_{i,2i+1}\, c_{2i+1}^{k+1} &&+ b_{i,2i+2}\, c_{2i+2}^{k+1} \\
&= \tfrac{1}{16}\, c_{2i-4}^{k+1} &&+ 0\, c_{2i-3}^{k+1} &&- \tfrac{9}{16}\, c_{2i-2}^{k+1} &&+ 1\, c_{2i-1}^{k+1} \\
&- \tfrac{9}{16}\, c_{2i}^{k+1} &&+ 0\, c_{2i+1}^{k+1} &&+ \tfrac{1}{16} c_{2i+2}^{k+1}
\end{aligned}
\tag{6.2.5}
$$

In matrix terms for a small cyclic system this would amount to:

$$
\mathbf{P}^{k+1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & -\frac{1}{16} \\
0 & 1 & 0 & 0 & 0 \\
-\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \\
0 & 0 & 0 & 1 & 0 \\
-\frac{1}{16} & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} \\
0 & 0 & 0 & 0 & 1 \\
\frac{9}{16} & -\frac{1}{16} & 0 & -\frac{1}{16} & \frac{9}{16}
\end{bmatrix}
\tag{6.2.6}
$$

$$
\mathbf{A}^{k+1} = \begin{bmatrix}
\frac{23}{32} & \frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{64} & 0 & \frac{1}{64} & 0 & -\frac{1}{8} & \frac{1}{4} \\
-\frac{1}{8} & \frac{1}{4} & \frac{23}{32} & \frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{64} & 0 & \frac{1}{64} & 0 \\
\frac{1}{64} & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{23}{32} & \frac{1}{4} & -\frac{1}{8} & 0 & \frac{1}{64} & 0 \\
\frac{1}{64} & 0 & \frac{1}{64} & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{23}{32} & \frac{1}{4} & -\frac{1}{8} & 0 \\
-\frac{1}{8} & 0 & \frac{1}{64} & 0 & \frac{1}{64} & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{23}{32} & \frac{1}{4}
\end{bmatrix}
\tag{6.2.7}
$$

$$
\mathbf{Q}^{k+1} = \begin{bmatrix}
-\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 \\
-\frac{1}{8} & \frac{23}{32} & -\frac{1}{8} & \frac{1}{64} & \frac{1}{64} \\
0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\
\frac{1}{64} & -\frac{1}{8} & \frac{23}{32} & -\frac{1}{8} & \frac{1}{64} \\
0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\
\frac{1}{64} & \frac{1}{64} & -\frac{1}{8} & \frac{23}{32} & -\frac{1}{8} \\
0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{8} & \frac{1}{64} & \frac{1}{64} & -\frac{1}{8} & \frac{23}{32} \\
-\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} \\
\frac{23}{32} & -\frac{1}{8} & \frac{1}{64} & \frac{1}{64} & -\frac{1}{8}
\end{bmatrix} \tag{6.2.8}
$$

$$
\mathbf{B}^{k+1} = \begin{bmatrix}
-\frac{9}{16} & 0 & \frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 & -\frac{9}{16} & 1 \\
-\frac{9}{16} & 1 & -\frac{9}{16} & 0 & \frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 \\
\frac{1}{16} & 0 & -\frac{9}{16} & 1 & -\frac{9}{16} & 0 & \frac{1}{16} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{16} & 0 & -\frac{9}{16} & 1 & -\frac{9}{16} & 0 & \frac{1}{16} & 0 \\
\frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 & -\frac{9}{16} & 1 & -\frac{9}{16} & 0
\end{bmatrix} \tag{6.2.9}
$$

The Gram matrix $\mho^{k+1}$ induced by this system of coefficients consists of the following two rows, repeated with a shift of two positions to the right on each repetition:

$$
\begin{bmatrix}
\cdots & 0 & \frac{1}{4096} & \frac{1}{256} & -\frac{33}{1024} & \frac{9}{256} & \frac{1}{16} & -\frac{53}{128} & \frac{2435}{2048} & -\frac{53}{128} & \frac{1}{16} & \frac{9}{256} & -\frac{33}{1024} & \frac{1}{256} & \frac{1}{4096} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \frac{1}{256} & 0 & \frac{9}{256} & \frac{1}{16} & -\frac{53}{128} & \frac{9}{8} & -\frac{53}{128} & \frac{1}{16} & \frac{9}{256} & 0 & \frac{1}{256} & 0 & \cdots
\end{bmatrix}
$$

The diagonal elements are $\frac{2435}{2048}$ and $\frac{9}{8}$.

7 Sample Applications

The Chaikin subdivision/reconstruction presented in Section 6 is so appealingly simple, that the sample applications given in Figures 9 and 10, which constitute our main examples, are given using this system. In order to carry out reverse subdivision and reconstruction on these examples, which use tensor-product data that, for the image data is not periodic and for the surface data is periodic in only one direction, we had to obtain a nonperiodic system for Chaikin subdivision, which we list below. Note that the sign patterns in $\mathbf{Q}^{k+1}$ and $\mathbf{B}^{k+1}$ have been adjusted. The wavelets and dual wavelets that the columns of $\mathbf{Q}^{k+1}$ and rows of $\mathbf{B}^{k+1}$ represent are not individually symmetric, but we have chosen a sign pattern that forms a symmetric set of wavelets on the closed bounded interval of their domain. Only the matrices for $\dim(\mathcal{V}^k) = 6$ and $\dim(\mathcal{V}^{k+1}) = 10$ are given. The expansion to larger dimensions should be obvious.

$$\mathbf{P}^{k+1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.1}$$

$$\mathbf{Q}^{k+1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & -\frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.2}$$

$$\mathbf{A}^{k+1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & \frac{3}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.3}$$

$$\mathbf{B}^{k+1} = \begin{bmatrix} -\frac{1}{2} & 1 & -\frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} & 1 & -\frac{1}{2} \end{bmatrix} \tag{7.4}$$

The Gram matrix corresponding to this system is:

$$\mho^{k+1} = \begin{bmatrix} \frac{3}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{4} & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{4} & -\frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{4} & \frac{5}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & \frac{5}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{3}{2} \end{bmatrix} \tag{7.5}$$

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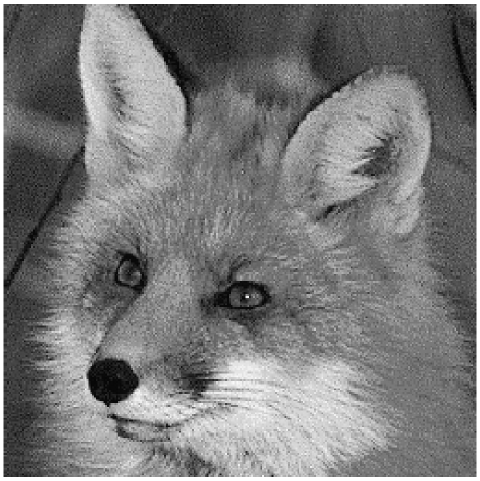

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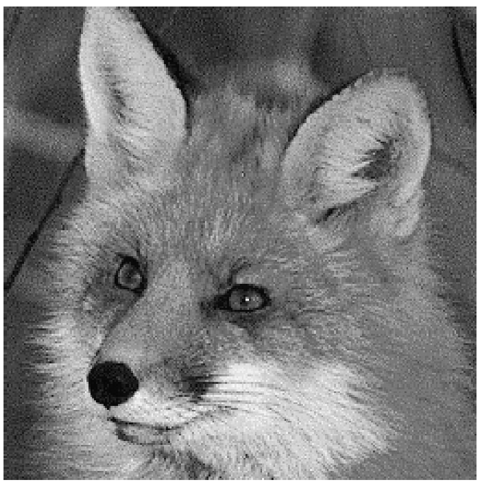
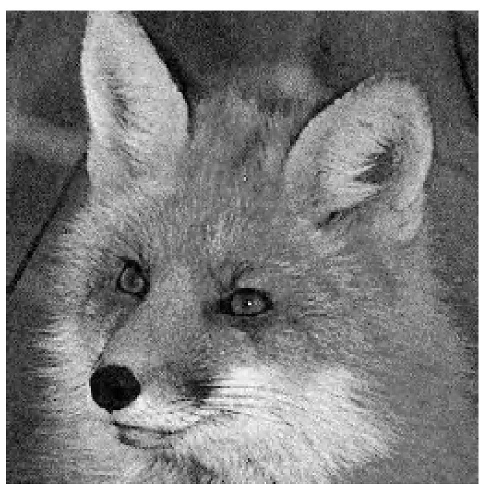
Coarse: 64 by 64



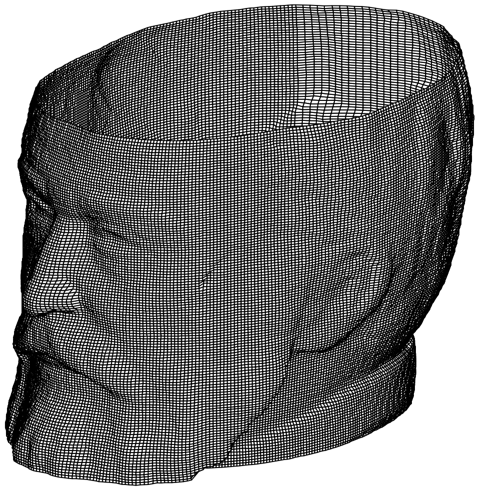
Reconstructed with C and D

Reconstructed via Subdivision Only

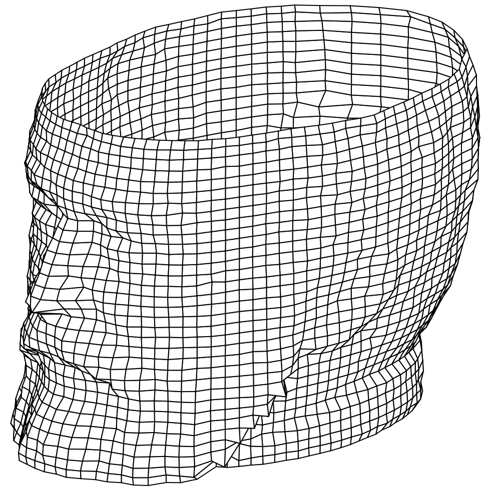


Reconstructed with Smallest 60% D Removed

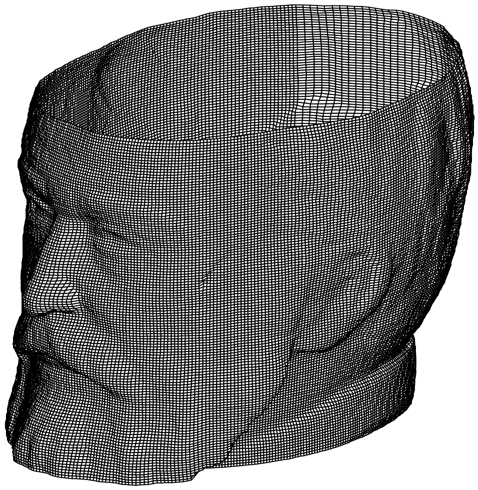
Figure 9: Fox Image



Original: 320 by 130



Coarse: 80 by 34



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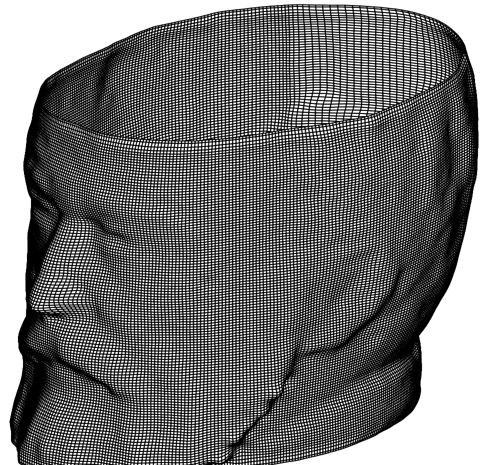

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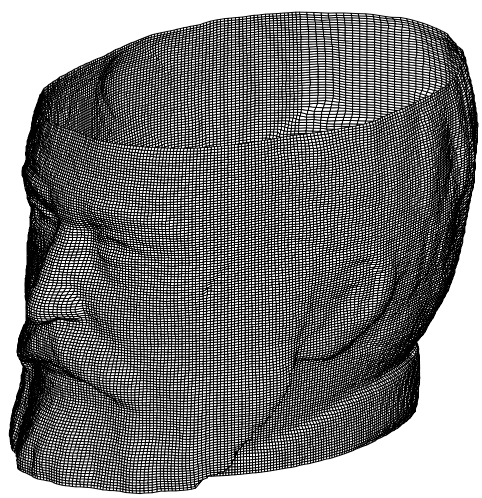

Reconstructed with Smallest 60% D Removed

Figure 10: Victor Hugo